

On the constants for some fractional Gagliardo-Nirenberg and Sobolev inequalities

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Abstract

We consider the inequalities of Gagliardo-Nirenberg and Sobolev in \mathbb{R}^d , formulated in terms of the Laplacian Δ and of the fractional powers $D^n := \sqrt{-\Delta}^n$ with real $n \geq 0$; we review known facts and present novel results in this area. After illustrating the equivalence between these two inequalities and the relations between the corresponding sharp constants and maximizers, we focus the attention on the \mathcal{L}^2 case where, for all sufficiently regular $f : \mathbb{R}^d \rightarrow \mathbb{C}$, the norm $\|D^j f\|_{\mathcal{L}^r}$ is bounded in terms of $\|f\|_{\mathcal{L}^2}$ and $\|D^n f\|_{\mathcal{L}^2}$, for $1/r = 1/2 - (\vartheta n - j)/d$, and suitable values of j, n, ϑ (with j, n possibly noninteger). In the special cases $\vartheta = 1$ and $\vartheta = j/n + d/2n$ (i.e., $r = +\infty$), related to previous results of Lieb and Ilyin, the sharp constants and the maximizers can be found explicitly; we point out that the maximizers can be expressed in terms of hypergeometric, Fox and Meijer functions. For the general \mathcal{L}^2 case, we present two kinds of upper bounds on the sharp constants: the first kind is suggested by the literature, the second one is an alternative proposal of ours, often more precise than the first one. We also derive two kinds of lower bounds. Combining all the available upper and lower bounds, the Gagliardo-Nirenberg and Sobolev sharp constants are confined to quite narrow intervals. Several examples are given.

Key words: fractional Gagliardo-Nirenberg and Sobolev inequalities.

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1 Introduction

In this paper we work in \mathbb{R}^d , using the Laplacian Δ , the operator $D := \sqrt{-\Delta}$ and its powers D^n with real exponent $n \geq 0$. In the sequel f stands for a complex-valued function on \mathbb{R}^d , with suitable regularity properties.

We consider the embedding inequalities of Gagliardo [12], Nirenberg [27] and Sobolev [32]. The terms *Gagliardo-Nirenberg inequality* and *Sobolev inequality* are used to indicate, respectively, the inequalities ⁽¹⁾

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^{\vartheta} \quad \left(\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q} - \frac{\vartheta n - j}{d} \right), \quad (1.1)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq S (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t} \quad (r \text{ as in (1.1)}) , \quad (1.2)$$

holding if the parameters p, q, j, n, ϑ, t fulfill appropriate conditions. Here and in the sequel, \mathcal{L}^p is the usual space $L^p \equiv L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1, +\infty)$, while \mathcal{L}^∞ is the subspace of $C(\mathbb{R}^d, \mathbb{C})$ made of the functions vanishing at infinity, with the usual sup norm (see the forthcoming Eq. (2.12), and the related comments).

The inequalities (1.1) (1.2) are found to be equivalent via appropriate scaling considerations. We are especially interested in their \mathcal{L}^2 versions which are obtained setting $p = q = t = 2$, and read

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad \left(\frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d} \right), \quad (1.3)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad (r \text{ as in (1.3)}) . \quad (1.4)$$

They hold under suitable conditions on j, n, ϑ , given in the forthcoming Eq. (5.9) and here anticipated:

$$0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad 0 \leq \vartheta n - j \leq \frac{d}{2}, \quad \vartheta \neq 1 \text{ if } n = j + \frac{d}{2}; \quad (1.5)$$

we write $G(j, n, \vartheta)$ and $S(j, n, \vartheta)$ for the sharp constants of (1.3) and (1.4), respectively. The aims of this paper are as follows.

- To summarize some basic facts related to Eqs. (1.1) (1.2), from the definition of the fractional Laplacian via Lizorkin distributions and Fourier transform to the derivation of the equivalence between (1.1) and (1.2) via scaling considerations. These generalities prepare the analysis of the \mathcal{L}^2 inequalities (1.3) (1.4).

¹The association of the cited authors to either (1.1) or (1.2) is to some extent conventional; in particular, the cited papers of Gagliardo consider mainly the inequality (1.2). However, these historical aspects are not relevant for our purposes.

- To review some special cases in which the sharp constants and some maximizers for either (1.3) or (1.4) have been determined in the literature; by the general equivalence mentioned before, any result of this kind for one of the two inequalities can be converted into a result for the other one.
- To point out a fact unnoticed in the literature, i.e., that the maximizers of the special cases mentioned before can be expressed in terms of hypergeometric functions, Fox H -functions or Meijer G -functions.
- To derive by more or less known strategies some upper bounds for the sharp constants of the general inequalities (1.3) and (1.4).
- To propose some lower bounds for these sharp constants and compare them with the above mentioned upper bounds. This analysis is performed in a fully quantitative way and ultimately confines the unknown sharp constants of (1.3) and (1.4) to rather small intervals, as shown by several numerical examples.

Let us describe with more details the organization of the paper. In Section 2 we fix our standards for some basic spaces of functions or distributions in \mathbb{R}^d ; moreover, we recall how to define the fractional Laplacians D^n via Fourier transform, in the framework of Lizorkin distributions.

In Section 3 (and in the related Appendix A) we present formally the Gagliardo-Nirenberg and Sobolev inequalities (1.1) (1.2) in their natural functional setting. Moreover, using some scaling considerations we derive the equivalence between (1.1) and (1.2), and make explicit the relations between the corresponding sharp constants and maximizers. This idea has been used in the literature in many special cases [2] [11]; our general formulation is contained in Proposition 3.8.

The results of Section 3 are stated on purely logical grounds, independently of the actual conditions on the parameters p, q, j, n, ϑ, t for the validity of (1.1) (1.2); these conditions are reported in Section 4, with an appropriate bibliographical support.

The remaining Sections 5, 6 (and the related Appendix B) form the longest part of the paper, entirely devoted to the \mathcal{L}^2 inequalities (1.3) (1.4); hereafter we describe in detail the contents of these sections that provide, amongst else, simple and self-consistent proofs of (1.3) or (1.4) (for j, n, ϑ as in Eq. (1.5)), independently of the general results on the validity of (1.1) (1.2).

- (i) In subsection 5.1 we begin our discussion of the \mathcal{L}^2 inequalities analyzing the elementary case $j = \vartheta n$, i.e., $r = 2$. In this case the Gagliardo-Nirenberg inequality (1.3) is reduced (via Fourier transform) to the Hölder inequality: the Gagliardo-Nirenberg sharp constant is readily found to be 1, and by the equivalence between (1.3) and (1.4) one also obtains the Sobolev sharp constant.

- (ii) In subsection 5.2 we pass to the (much less elementary) case $\vartheta = 1$. Eq. (1.3) (and, more generally, Eq. (1.1)) with $\vartheta = 1$ is equivalent to the so-called Hardy-Littlewood-Sobolev inequality concerning convolution with a power of the radius $|x|$ ($x \in \mathbb{R}^d$) [17] [32] (see, e.g., [25] for a more up-to-date reference on these issues). The sharp constant and the maximizers of the Hardy-Littlewood-Sobolev inequality in the \mathcal{L}^2 case have been found by Lieb [19], and these results can be used in relation to the $\vartheta = 1$ case of (1.3); this was done in explicit terms in [7] (with the additional assumption $j = 0$). A review of these facts is presented (extending the considerations of [7] to nonzero values of j). The maximizer derived along these lines is a radial function and can be represented as the inverse Fourier transforms of a Bessel-type function; when expressed in terms of the ordinary space variables, it is an elementary function for $j = 0$ and a hypergeometric-type function for suitable values of j, n .
- (iii) In subsection 5.3, following a path indicated by [16], we combine the previous results on the cases $j = \vartheta n$ and $\vartheta = 1$ to derive the Gagliardo-Nirenberg inequality (1.3) and to obtain upper bounds $G_+(j, n, \vartheta)$ on its sharp constants in an “almost general case”; this covers all choices of j, n, ϑ in Eq. (1.5) except $\vartheta = j/n + d/2n$ (i.e., $r = +\infty$). The upper bounds obtained by this strategy coincide with the sharp constant if $\vartheta = 1$, and diverge if ϑ approaches the exceptional value $j/n + d/2n$. By the equivalence between the Gagliardo-Nirenberg and the Sobolev inequalities, the above results can be rephrased making reference to (1.4); in particular we have upper bounds $S_+(j, n, \vartheta)$ on the corresponding sharp constants (again, coinciding with the sharp constant for $\vartheta = 1$ and diverging for $\vartheta \rightarrow j/n + d/2n$).
- (iv) Subsection 5.4 focuses on the previously excluded case $\vartheta = j/n + d/2n$ (i.e., $r = +\infty$), requiring a different strategy. For $j = 0$ this case was analyzed by Ilyin [18] who derived via Fourier transform the Sobolev inequality (1.4), found the sharp constant and the maximizer, and then pointed out the implications of these results on the Gagliardo-Nirenberg inequality (1.3). In this subsection we propose a similar derivation for (1.4), giving the sharp constant and a maximizer holding for arbitrary j ; the consequences for (1.3) are indicated. Our maximizer is radial and agrees for $j = 0$ with the one of Ilyin; as in [18], it can be represented as the inverse Fourier transform of an elementary function. We also derive its expressions in terms of the space variables, using the Fox H -function or the Meijer G -function (whose definitions are reviewed in Appendix B).
- (v) In subsection 5.5 we propose a second, “almost general” approach to the \mathcal{L}^2 inequalities (1.3) (1.4), alternative to the one of item (iii); this covers all choices of j, n, ϑ in Eq. (1.5), except the special case $\vartheta = 1$ of item (ii). We use

again Fourier transform arguments, inspired by a previous work of ours [26] on a variant of (1.4); these yield an alternative proof of the Sobolev inequality (1.4), giving upper bounds $S_{++}(j, n, \vartheta)$ for its sharp constants. These bounds coincide with the Sobolev sharp constant for $\vartheta = j/n + d/2n$, and diverge for $\vartheta \rightarrow 1$. Due to the equivalence between the Sobolev and Gagliardo-Nirenberg inequalities, these results on (1.4) yield upper bounds $G_{++}(j, n, \vartheta)$ on the sharp constants of (1.3), with an analogous behavior for $\vartheta = j/n + d/2n$ and for $\vartheta \rightarrow 1$.

- (vi) As emphasized in subsection 5.6, the upper bounds G_+, S_+ of item (iii) and G_{++}, S_{++} of item (v) cover jointly the general \mathcal{L}^2 inequalities (1.3) (1.4). The upper bounds G_+, S_+ are expected to be better (i.e., smaller) for ϑ close to 1, while G_{++}, S_{++} are expected to be better for ϑ close to $j/n + d/2n$. To be more specific one can make a direct comparison of the numerical values of these bounds, an issue that is treated in Section 6 for $d = 1, 2, 3$ and some test values of j, n, ϑ .
- (vii) As a final step in the theoretical investigation of the \mathcal{L}^2 inequalities (1.3) (1.4), in subsection 5.7 we derive lower bounds for their sharp constants. We obtain two types of lower bounds $G_-(j, n, \vartheta)$, $S_-(j, n, \vartheta)$ and $G_{--}(j, n, \vartheta)$, $S_{--}(j, n, \vartheta)$, derived substituting “trial functions” of two kinds for f in (1.3) or in (1.4). The $-$ bounds hold under the general \mathcal{L}^2 conditions (1.5), while the $--$ bounds require some limitations for j, n, ϑ (see Eq. (5.110)).
- (viii) In Section 6, as examples we write explicitly the sharp constants and maximizers of (1.3) (1.4) for $d = 1, 2, 3$, $\vartheta = j/n + d/2n$ (see item (iv)) and some choices of j, n : see Tables I, II. Moreover, we give the numerical values of the upper bounds G_+, G_{++} and of the lower bounds G_-, G_{--} on the sharp constants of (1.3), for $d = 1, 2, 3$ and several choices of j, n, ϑ : see Table III. (In these examples the $+$ bounds are better than the $++$ bounds for $\vartheta \simeq 1$, as expected; however, this occurs only for ϑ very close to 1.) In these numerical tests the best lower and upper bounds are close together, thus confining the sharp constants $G(j, n, \vartheta)$ to narrow intervals. By the equivalence between the Gagliardo-Nirenberg and the Sobolev inequality, it would be easy to produce similar numerical results for the bounds S_+, S_{++}, S_-, S_{--} on the sharp constants of (1.4).

In the above description of the contents of the paper, we have indicated their connections with the existing literature to the best of our knowledge; more details are given in the sequel. We hope that the survey of known results presented here in a unified language, and our contributions mentioned in items (v)-(viii), will allow a more complete understanding of the \mathcal{L}^2 Gagliardo-Nirenberg-Sobolev inequalities.

2 Some preliminaries

Notations. Throughout the paper we intend

$$0^0 := (+\infty)^0 := 1 ; \quad (2.1)$$

$$(a^t + b^t)^{1/t} := \max(a, b) \quad \text{for } a, b \in [0, +\infty) \text{ and } t = +\infty . \quad (2.2)$$

We work in the Euclidean space \mathbb{R}^d for a fixed space dimension $d \in \{1, 2, 3, \dots\}$. We often write \mathbf{x} for the identity map $\mathbb{R}^d \rightarrow \mathbb{R}^d$, $x \mapsto x$, and $|\mathbf{x}|$ for the map $\mathbb{R}^d \rightarrow [0, +\infty)$, $x \mapsto |x|$. When dealing with the Fourier transform, a typical wave vector in \mathbb{R}^d is indicated with k ; in this framework, the identity map and the Euclidean norm of \mathbb{R}^d are indicated with \mathbf{k} and $|\mathbf{k}|$.

Given two complex topological vector spaces E, F we say that E is continuously embedded in F , and write $E \hookrightarrow F$, if E is a vector subspace of F and the natural inclusion of E into F is continuous.

Standard terminology for Banach space inequalities. Let us consider two complex Banach spaces X, Y with norms $\|\cdot\|_X, \|\cdot\|_Y$. Moreover, assume we are given a linear map $\mathfrak{F} : X \rightarrow Y'$ where Y' is vector space containing Y as a vector subspace. We often consider statements with the following structure:

$$\mathfrak{F}X \subset Y , \quad \|\mathfrak{F}f\|_Y \leq C\|f\|_X \quad \text{for all } f \in X \text{ and some } C \in [0, +\infty). \quad (2.3)$$

Any such statement is referred to as an *inequality*; of course (2.3) indicates that \mathfrak{F} is continuous from X to Y and that, if \mathfrak{F} is the identity, there is a continuous embedding $X \hookrightarrow Y$. The *sharp constant* C_s for the inequality (2.3) is the inf of the set of the constants $C \in [0, +\infty)$ which fulfill it; this inf is in fact a minimum. A *maximizer* for (2.3), if it exists, is a nonzero element $f \in X$ such that $\|\mathfrak{F}f\|_Y = C_s\|f\|_X$.

It is clear that (2.3) holds if and only if $\mathfrak{F}X \subset Y$ and the ratio $\|\mathfrak{F}f\|_Y/\|f\|_X$ is bounded for f ranging in $X \setminus \{0\}$. If this happens, the sharp constant can be expressed as $C_s = \sup_{f \in X, f \neq 0} \|\mathfrak{F}f\|_Y/\|f\|_X$; an element $f \in X \setminus \{0\}$ is a maximizer for (2.3) if and only if it is a maximum point for the above ratio. Of course, any transformation leaving invariant this ratio sends maximizers into maximizers; the simplest example is the map $f \mapsto Kf$ where $K \in \mathbb{C} \setminus \{0\}$.

Some spaces of functions and distributions on \mathbb{R}^d . As usual, we say that a function $\phi : \mathbb{R}^d \rightarrow \mathbb{C}$ is rapidly decreasing if $(1 + |\mathbf{x}|)^N \phi$ is bounded for all $N \in \mathbb{N}$.

We employ the standard symbol $\mathcal{S}(\mathbb{R}^d, \mathbb{C}) \equiv \mathcal{S}$ for the Schwartz space, formed by the C^∞ functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$ rapidly decreasing with all their derivatives; this space is equipped with the Fréchet topology induced by the seminorms $p_{N, i_1, \dots, i_m}(\varphi) := \sup_{x \in \mathbb{R}^d} (1 + |x|)^N |\partial_{i_1, \dots, i_m} \varphi(x)|$, where $N, m \in \mathbb{N}$ and $i_1, \dots, i_m \in \{1, \dots, d\}$.

We consider the Fourier transform

$$\mathcal{F} : \mathcal{S} \rightarrow \mathcal{S} , \quad \varphi \mapsto \mathcal{F}\varphi , \quad (2.4)$$

normalized so that $\mathcal{F}\varphi(k) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} dx e^{-ik \bullet x} \varphi(x)$; this is a linear homeomorphism of \mathcal{S} into itself. The dual space $\mathcal{S}'(\mathbb{R}^d, \mathbb{C}) \equiv \mathcal{S}'$ is the standard space of tempered distributions, and one can extend the Fourier transform to a map of \mathcal{S}' into itself.

Some less conventional spaces of distributions and the related Fourier transforms are more interesting in relation to fractional differential calculus [31]. Their construction relies on the Lizorkin-type spaces of test functions

$$\Phi(\mathbb{R}^d, \mathbb{C}) \equiv \Phi := \{ \phi \in \mathcal{S} \mid \int_{\mathbb{R}^d} dx x_{i_1} \dots x_{i_m} \phi(x) = 0 \text{ for all } i_1, \dots, i_m \} , \quad (2.5)$$

$$\Psi(\mathbb{R}^d, \mathbb{C}) \equiv \Psi := \{ \psi \in \mathcal{S} \mid \partial_{i_1, \dots, i_m} \psi(0) = 0 \text{ for all } i_1, \dots, i_m \} . \quad (2.6)$$

(In both cases, “for all i_1, \dots, i_m ” means “for all $m \in \mathbb{N}$ and $i_1, \dots, i_m \in \{1, \dots, d\}$ ”. Obviously enough, $x_{i_1} \dots x_{i_m} := 1$ if $m = 0$. Functions $\psi \in \Psi$ are usually written as $\psi : \mathbb{R}^d \rightarrow \mathbb{C}$, $k \mapsto \psi(k)$.)

Φ, Ψ are closed vector subspaces of \mathcal{S} , and thus are Fréchet spaces with the induced topology. One readily checks that

$$\mathcal{F}\Phi = \Psi , \quad (2.7)$$

and that \mathcal{F} is a linear homeomorphism between Φ and Ψ . The Lizorkin-type distribution spaces $\Phi'(\mathbb{R}^d) \equiv \Phi'$ and $\Psi'(\mathbb{R}^d) \equiv \Psi'$ are the dual spaces of Φ and Ψ , equipped with their weak topologies ⁽²⁾. One can define a Fourier transform

$$\mathcal{F} : \Phi' \rightarrow \Psi', \quad \langle \mathcal{F}f, \psi \rangle := \langle f, \mathcal{F}^{-1}\psi_R \rangle \quad \text{for } f \in \Phi', \psi \in \Psi, \psi_R(k) := \psi(-k) ; \quad (2.8)$$

this is a linear homeomorphism between Φ' and Ψ' ⁽³⁾.

For $j \in \{1, \dots, d\}$ the map $\phi \mapsto \partial_j \phi$ is linear and continuous from Φ into itself; we define the distributional derivative $\partial_j : \Phi' \rightarrow \Phi'$ by the usual procedure *à la Schwartz*, i.e., setting $\langle \partial_j f, \phi \rangle := -\langle f, \partial_j \phi \rangle$ for all $f \in \Phi', \phi \in \Phi$.

To go on, let us consider any function $\eta \in C^\infty(\mathbb{R}^d \setminus \{0\}, \mathbb{C})$ such that, for all $n \in \mathbb{N}$ and $j_1, \dots, j_n \in \{1, \dots, d\}$ one has $|\partial_{j_1, \dots, j_n} \eta(k)| = O(|k|^{-M})$ for $k \rightarrow 0$ and $|\partial_{j_1, \dots, j_n} \eta(k)| = O(|k|^N)$ for $k \rightarrow \infty$, for suitable real exponents N, M (depending on n, j_1, \dots, j_n). For $\psi \in \Psi$ the product $\eta\psi$ is clearly C^∞ on $\mathbb{R}^d \setminus \{0\}$ and possesses a

²Of course, there are continuous linear maps $\mathcal{S}' \rightarrow \Phi', f \mapsto f \upharpoonright \Phi$ and $\mathcal{S}' \rightarrow \Psi', f \mapsto f \upharpoonright \Psi$; the kernels of these maps are, respectively, the space $Pol(\mathbb{R}^d) \equiv Pol$ of the polynomial functions on \mathbb{R}^d and the space $Dir(\mathbb{R}^d) \equiv Dir$ of the finite linear combinations of the Dirac delta at the origin and of its derivatives. Therefore, there are linear homeomorphisms $\mathcal{S}'/Pol \rightarrow \Phi'$ and $\mathcal{S}'/Dir \rightarrow \Psi'$ [31].

³Formally, one has $\mathcal{F}^{-1}\psi_R = \mathcal{F}\psi$. This identity, even though correct, is somehow misleading; it can be stated just because the spaces of wave vectors k and position vectors x are both identified with \mathbb{R}^d .

unique C^∞ extension to \mathbb{R}^d , vanishing at the origin and rapidly decreasing with all its derivatives: in this sense, we have $\eta\psi \in \Psi$. The map

$$\Psi \rightarrow \Psi, \quad \psi \mapsto \eta\psi \quad (2.9)$$

is linear and continuous. This fact can be used to introduce a continuous linear map

$$\Psi' \rightarrow \Psi', \quad g \mapsto \eta g \quad (2.10)$$

putting $\langle \eta g, \psi \rangle := \langle g, \eta\psi \rangle$ for all $\psi \in \Psi$. *A fortiori*, this construction works in the special case of a function $\eta \in C^\infty(\mathbb{R}^d, \mathbb{C})$ with the previously mentioned behavior for $k \rightarrow \infty$. In particular, let us choose for η the function $\mathbf{k}_j : \mathbb{R}^d \rightarrow \mathbb{R}$, $k \mapsto k_j$, for some $j \in \{1, \dots, d\}$. This gives a map $\Psi' \rightarrow \Psi'$, $g \mapsto \mathbf{k}_j g$ and it turns out that $\mathcal{F}(\partial_j f) = i\mathbf{k}_j \mathcal{F}f$ for all $f \in \Phi'$.

For each $p \in [1, +\infty]$, we write L^p for the usual space $L^p(\mathbb{R}^d, \mathbb{C})$ and $\|\cdot\|_{L^p}$ for its norm; moreover, we define as follows the Banach space \mathcal{L}^p and its norm:

$$\mathcal{L}^p := L^p, \quad \|\cdot\|_{\mathcal{L}^p} := \|\cdot\|_{L^p} \text{ if } p \in [1, +\infty), \quad (2.11)$$

$$\mathcal{L}^\infty := C_0(\mathbb{R}^d, \mathbb{C}) = \{f \in C(\mathbb{R}^d, \mathbb{C}) \mid \lim_{x \rightarrow \infty} f(x) = 0\}, \quad \|f\|_{\mathcal{L}^\infty} := \sup_{x \in \mathbb{R}^d} |f(x)|. \quad (2.12)$$

\mathcal{L}^∞ is a closed subspace of L^∞ , and $\|\cdot\|_{\mathcal{L}^\infty}$ is just the restriction to this subspace of $\|\cdot\|_{L^\infty}$ ⁽⁴⁾. For any $p \in [1, +\infty]$, we clearly have

$$\Phi, \Psi \hookrightarrow \mathcal{L}^p. \quad (2.13)$$

Again for $p \in [1, +\infty]$, given $f \in \mathcal{L}^p$ we can define a continuous linear form $\langle f, \cdot \rangle$ on Φ setting $\langle f, \phi \rangle := \int_{\mathbb{R}^d} dx f(x)\phi(x)$ for $\phi \in \Phi$. The map $f \in \mathcal{L}^p \mapsto \langle f, \cdot \rangle \in \Phi'$ is a continuous linear injection ⁽⁵⁾; thus, up to a natural identification (to be employed from now on), for all $p \in [1, +\infty]$ we have

$$\mathcal{L}^p \hookrightarrow \Phi'. \quad (2.14)$$

In a similar way, for $p \in [1, +\infty]$ and $f \in \mathcal{L}^p$ we can define a continuous linear form on Ψ setting $\langle f, \psi \rangle := \int_{\mathbb{R}^d} dk f(k)\psi(k)$ for all $\psi \in \Psi$. We have a continuous linear injection $f \in \mathcal{L}^p \mapsto \langle f, \cdot \rangle \in \Psi'$, so that

$$\mathcal{L}^p \hookrightarrow \Psi'. \quad (2.15)$$

⁴Considering the family (2.11) (2.12) is not unusual, see e.g. [8] [9] where similar families are employed in different situations; admittedly, the notation \mathcal{L}^p is not standard. For the purposes of the present work, the space \mathcal{L}^∞ is much more interesting than L^∞ .

⁵According to the footnote before Eq. (2.8), the kernel of this map is the intersection of \mathcal{L}^p with the space Pol of polynomials on \mathbb{R}^d ; clearly, $\mathcal{L}^p \cap Pol = \{0\}$ for all $p \in [1, +\infty]$. To obtain this result for $p = \infty$, it is crucial to define \mathcal{L}^∞ in terms of functions vanishing at infinity.

Of course, many other topological vector spaces of complex, measurable functions on \mathbb{R}^d can be continuously embedded in Φ' or Ψ' using the previous prescriptions to identify a function f with a continuous linear form on Φ or Ψ . For example, we have the embeddings

$$\Phi \hookrightarrow \Phi', \quad \Psi \hookrightarrow \Psi' \quad (2.16)$$

(which can be seen as trivial consequences of Eqs. (2.13)-(2.15); using the second of these relations, we can view the map (2.9) as a restriction of the map (2.10)).

Hausdorff-Young inequality. Let

$$p \in [1, 2], \quad p' \text{ such that } 1/p + 1/p' = 1; \quad (2.17)$$

it is well known that ⁽⁶⁾

$$f \in \mathcal{L}^p \Rightarrow \mathcal{F}f \in \mathcal{L}^{p'}, \|\mathcal{F}f\|_{\mathcal{L}^{p'}} \leq C_p \|f\|_{\mathcal{L}^p}, \quad C_p := \frac{1}{(2\pi)^{d/p-d/2}} \left[\frac{(1/p')^{1/p'}}{(1/p)^{1/p}} \right]^{d/2}. \quad (2.18)$$

The inequality in (2.18) is the familiar *Hausdorff-Young inequality*; the constant $C_p \equiv C_{pd}$ is known to be sharp (see [20], [21] Chapter 5 and references therein; our expression for C_p differs by a factor from the one in [21] due to a different normalization for the Fourier transform). A Hausdorff-Young inequality holds as well for the inverse Fourier transform: with p, p' as in (2.17) and C_p as in (2.18),

$$g \in \mathcal{L}^p \Rightarrow \mathcal{F}^{-1}g \in \mathcal{L}^{p'}, \|\mathcal{F}^{-1}g\|_{\mathcal{L}^{p'}} \leq C_p \|g\|_{\mathcal{L}^p}; \quad (2.19)$$

the constant C_p of (2.18) is sharp as well for the formulation (2.19).

Fractional Laplacian. Let $s \in \mathbb{R}$, and consider the C^∞ function $|\mathbf{k}|^s : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ (having a C^∞ extension to \mathbb{R}^d if and only if s is a nonnegative, even integer). Making reference to Eqs. (2.9) (2.10) and to the related comments, we can define two maps

$$\Psi \rightarrow \Psi, \quad \psi \mapsto |\mathbf{k}|^s \psi; \quad \Psi' \rightarrow \Psi', \quad g \mapsto |\mathbf{k}|^s g; \quad (2.20)$$

both of them are linear homeomorphisms, with inverses corresponding to multiplication by $|\mathbf{k}|^{-s}$.

Composing the second map (2.20) with $\mathcal{F}^{-1} : \Psi' \rightarrow \Phi'$ and $\mathcal{F} : \Phi' \rightarrow \Psi'$ we obtain the linear homeomorphism

$$D^s : \Phi' \rightarrow \Phi', \quad f \mapsto D^s f := \mathcal{F}^{-1}(|\mathbf{k}|^s \mathcal{F}f), \quad (2.21)$$

⁶Note that (2.18) holds as well for $p = 1$, $p' = +\infty$, using our definition (2.12) for \mathcal{L}^∞ ; in fact, according to the Riemann-Lebesgue lemma [21], the Fourier transform of an \mathcal{L}^1 function is a continuous function vanishing at infinity.

with inverse D^{-s} ; of course, $D^s D^t = D^{s+t}$ for any real s, t . Taking into account Eqs. (2.7) (2.20) and the comments which accompany them, we see that

$$D^s \Phi = \Phi \quad (2.22)$$

(identifying Φ with a subspace of Φ' , see (2.16)) and that D^s is a homeomorphism of Φ into itself. Noting that the usual Laplacian Δ fulfills $-\Delta f = \mathcal{F}^{-1}(|\mathbf{k}|^2 \mathcal{F}f)$, we see that there would be good reasons to write

$$D^s \equiv \sqrt{-\Delta}^s ; \quad (2.23)$$

however, the symbol $\sqrt{-\Delta}^s$ is never employed in the sequel.

Convolution, and representation of D^{-n} for $n \in (0, d)$. Given two measurable functions $g, f : \mathbb{R}^d \rightarrow \mathbb{C}$, the convolution $g * f : \mathbb{R}^d \rightarrow \mathbb{C}$, $(g * f)(x) := \int_{\mathbb{R}^d} dy g(x - y) f(y)$ can be defined if the previous integral exists for almost all x . This happens, in particular, if $g := 1/|\mathbf{x}|^{d-n}$ and $f \in \mathcal{L}^q$ with $0 < n < d$ and $1 \leq q < d/n$ (see [33], Chapter V, Theorem 1). For n, q and f as above, one has

$$D^{-n} f = \frac{Z_n}{|\mathbf{x}|^{d-n}} * f , \quad Z_n \equiv Z_{dn} := \frac{\Gamma(d/2 - n/2)}{2^n \pi^{d/2} \Gamma(n/2)} . \quad (2.24)$$

(see again [33], Chapter V, Lemma 1 or [25], Chapters 2 and 7) (⁷).

Translations and scalings. Let $a \in \mathbb{R}^d$, $\lambda \in (0, +\infty)$; we consider the translation $\mathbb{R}^d \rightarrow \mathbb{R}^d$, $x \mapsto x + a$ and the scaling transformation $\mathbb{R}^d \rightarrow \mathbb{R}^d$, $x \mapsto \lambda x$.

If ϕ is a complex-valued function on \mathbb{R}^d , the a -translated function ϕ^a and the λ -rescaled function ϕ_λ are the compositions of ϕ with the previously mentioned transformations; thus

$$\phi^a, \phi_\lambda : \mathbb{R}^d \rightarrow \mathbb{C} , \quad \phi^a(x) := \phi(x + a), \quad \phi_\lambda(x) := \phi(\lambda x) . \quad (2.25)$$

One proves that $\phi \in \Phi \Rightarrow \phi^a, \phi_\lambda \in \Phi$. For $f \in \Phi'$, $a \in \mathbb{R}^d$ and $\lambda \in (0, +\infty)$ we define $f^a, f_\lambda \in \Phi'$ by

$$\langle f^a, \phi \rangle := \langle f, \phi^{-a} \rangle , \quad \langle f_\lambda, \phi \rangle := \lambda^{-d} \langle f, \phi_{\lambda^{-1}} \rangle \quad \text{for all } \phi \in \Phi' . \quad (2.26)$$

⁷The proof of (2.24) can be sketched as follows. By definition $D^{-n} f = \mathcal{F}^{-1}(|\mathbf{k}|^{-n} \mathcal{F}f)$. But $\mathcal{F}^{-1}(uv) = (2\pi)^{-d/2} \mathcal{F}^{-1}u * \mathcal{F}^{-1}v$ under suitable conditions on the functions u, v ; applying this with $u = |\mathbf{k}|^{-n}$ and $v = \mathcal{F}f$ one gets $D^{-n} f = (2\pi)^{-d/2} (\mathcal{F}^{-1}|\mathbf{k}|^{-n}) * f$. Finally, one shows that $(2\pi)^{-d/2} \mathcal{F}^{-1}|\mathbf{k}|^{-n} = Z_n |\mathbf{x}|^{-d+n}$. (These manipulations are related to the following prescriptions: for each $s \in \mathbb{R}$ and $a \in (-d, 0)$, the functions $|\mathbf{k}|^s$ and $|\mathbf{x}|^a$ are identified with elements of Ψ' and Φ' , respectively, setting $\langle |\mathbf{k}|^s, \psi \rangle := \int_{\mathbb{R}^d} dk |k|^s \psi(k)$ and $\langle |\mathbf{x}|^a, \phi \rangle := \int_{\mathbb{R}^d} dx |x|^a \phi(x)$. The conditions on s, a and the features of the test functions spaces Ψ, Φ ensure that the previous integrals converge and depend continuously on $\phi \in \Phi, \psi \in \Psi$).

Let $s \in \mathbb{R}$, $p \in [1, +\infty]$. Clearly, the operators D^s and the spaces \mathcal{L}^p are translation invariant ($f \in \Phi' \Rightarrow D^s f^a = (D^s f)^a$; $f \in \mathcal{L}^p \Rightarrow f^a \in \mathcal{L}^p$, $\|f^a\|_{\mathcal{L}^p} = \|f\|_{\mathcal{L}^p}$). As for the scaling by $\lambda \in (0, +\infty)$, the following is proved by elementary means:

$$f \in \Phi' \Rightarrow D^s f_\lambda = \lambda^s (D^s f)_\lambda ; \quad (2.27)$$

$$f \in \mathcal{L}^p \Rightarrow f_\lambda \in \mathcal{L}^p, \|f_\lambda\|_{\mathcal{L}^p} = \lambda^{-d/p} \|f\|_{\mathcal{L}^p} ; \quad (2.28)$$

$$f \in \Phi', D^s f \in \mathcal{L}^p \Rightarrow D^s f_\lambda \in \mathcal{L}^p, \|D^s f_\lambda\|_{\mathcal{L}^p} = \lambda^{s-d/p} \|D^s f\|_{\mathcal{L}^p} \quad (2.29)$$

(of course, the third statement is a consequence of the first two).

Spaces of Riesz potentials. Let $q \in [1, +\infty]$, $n \in [0, +\infty)$. The *space of Riesz potentials* (or *homogeneous Sobolev space*) of type q, n is ⁽⁸⁾

$$\dot{\mathcal{L}}^{q,n}(\mathbb{R}^d) \equiv \dot{\mathcal{L}}^{q,n} := \{f \in \Phi' \mid D^n f \in \mathcal{L}^q\} . \quad (2.30)$$

By construction, we have a linear isomorphism $\dot{\mathcal{L}}^{q,n} \rightarrow \mathcal{L}^q$, $f \mapsto D^n f$ with inverse $\mathcal{L}^q \rightarrow \dot{\mathcal{L}}^{q,n}$, $h \mapsto D^{-n} h$. Due to this fact, $\dot{\mathcal{L}}^{q,n}$ is a Banach space with norm

$$\|f\|_{\dot{\mathcal{L}}^{q,n}} := \|D^n f\|_{\mathcal{L}^q} . \quad (2.31)$$

Inhomogeneous Sobolev spaces. Let $p, q \in [1, +\infty]$, $n \in [0, +\infty)$. The *inhomogeneous Sobolev space* of order p, q, n is

$$\mathcal{L}^{p,q,n}(\mathbb{R}^d) \equiv \mathcal{L}^{p,q,n} := \mathcal{L}^p \cap \dot{\mathcal{L}}^{q,n} = \{f \in \mathcal{L}^p \mid D^n f \in \mathcal{L}^q\} . \quad (2.32)$$

This is a Banach space with respect to any one of the equivalent norms

$$\|f\|_{\mathcal{L}^{p,q,n}|t} := (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t} \quad (t \in [1, +\infty]) \quad (2.33)$$

(if $t = +\infty$, this definition must be understood following (2.2)) ⁽⁹⁾.

Embedding and density statements. Let $p, q \in [1, +\infty]$, $n \in [0, +\infty)$. It is evident that $\Phi \hookrightarrow \mathcal{L}^{p,q,n} \hookrightarrow \dot{\mathcal{L}}^{q,n}$. Moreover Φ (and, consequently, $\mathcal{L}^{p,q,n}$) is dense in $\dot{\mathcal{L}}^{q,n}$ if $q \neq 1, +\infty$ ⁽¹⁰⁾.

⁸Most treatments of Riesz potential spaces do not consider the case $q = +\infty$, see e.g. [14] [31]; the same comment can be done on the inhomogeneous Sobolev spaces introduced hereafter. $\dot{\mathcal{L}}^{q,n}$ is often denoted with the alternative notation $I^n(\mathcal{L}^q)$, in which I^n stands for D^{-n} , see [31].

⁹The fact that $\mathcal{L}^{p,q,n}$ is a vector space and the equivalence of all norms $\|\cdot\|_{\mathcal{L}^{p,q,n}|t}$ are evident. To prove completeness, let us consider a Cauchy sequence $(f_\ell)_{\ell \in \mathbb{N}}$ in $\mathcal{L}^{p,q,n}$. Then $(f_\ell)_{\ell \in \mathbb{N}}$ and $(D^n f_\ell)_{\ell \in \mathbb{N}}$ are Cauchy sequences in \mathcal{L}^p and \mathcal{L}^q , respectively, so there are functions $g \in \mathcal{L}^p$ and $h \in \mathcal{L}^q$ such that $f_\ell \rightarrow g$ in \mathcal{L}^p and $D^n f_\ell \rightarrow h$ in \mathcal{L}^q . But $\mathcal{L}^p, \mathcal{L}^q \hookrightarrow \Phi'$, so $f_\ell \rightarrow g$ and $D^n f_\ell \rightarrow h$ in Φ' ; by the continuity of D^{-n} on Φ' we also infer $f_\ell \rightarrow D^{-n} h$ in Φ' so that $D^{-n} h = g$, i.e., $h = D^n g$. In conclusion $f_\ell \rightarrow g$ in \mathcal{L}^p and $D^n f_\ell \rightarrow D^n g$ in \mathcal{L}^q , which means $f_\ell \rightarrow g$ in $\mathcal{L}^{p,q,n}$.

¹⁰To prove this density statement, we consider an $f \in \dot{\mathcal{L}}^{q,n}$ and show that f is the limit of a sequence of functions $f_\ell \in \Phi$. Indeed, $D^n f \in \mathcal{L}^q$ and it is known that Φ is dense in \mathcal{L}^q (see [31] page 41, Theorem 2.7), so there is a sequence $(g_\ell)_{\ell \in \mathbb{N}}$ in Φ such that $g_\ell \rightarrow D^n f$ in \mathcal{L}^q . Recalling that D^n is a one-to-one map of Φ into itself, with inverse D^{-n} , let us introduce the functions $f_\ell := D^{-n} g_\ell \in \Phi$; then by construction $D^n f_\ell = g_\ell \rightarrow D^n f$ in \mathcal{L}^q , which means $f_\ell \rightarrow f$ in $\dot{\mathcal{L}}^{q,n}$.

3 The inequalities of Gagliardo-Nirenberg and Sobolev, with their logical connections

The aims of this section are:

- (i) to *define* in formal terms the above inequalities;
- (ii) to point out their *logical connections* (i.e., the fact that one of them implies the other one).

This discussion is carried over independently of the *validity conditions* for the above inequalities, that are the subject of Section 4. We think that the logical status of these inequalities has its own interest, independently of the strategies that one can use to prove their validity in more or less general situations.

3.1 The Gagliardo-Nirenberg inequality

The definition given hereafter for this inequality involves certain parameters p, q, j, n, ϑ ; for the moment we put on the parameters the minimal conditions ensuring well definedness of both sides in the inequality and certain scaling properties of general use. The *validity* of the inequality requires more stringent conditions, see Section 4.

The minimal conditions are the following ones:

$$1 \leq p, q \leq +\infty, \quad 0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad (3.1)$$

$$0 \leq \frac{1-\vartheta}{p} + \frac{\vartheta}{q} - \frac{\vartheta n - j}{d} \leq 1;$$

due to the last condition, there is a unique $r \equiv r(p, q; j, n, \vartheta) \in [1, +\infty]$ such that

$$\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q} - \frac{\vartheta n - j}{d}. \quad (3.2)$$

3.1 Definition. Let p, q, j, n, ϑ, r be as in (3.1) (3.2). The Gagliardo-Nirenberg inequality of order $(p, q; j, n, \vartheta)$ is the following statement:

$$\mathcal{L}^{p,q,n} \subset \mathcal{L}^{p,r,j} \text{ and} \quad (3.3)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^{\vartheta} \quad \text{for some } G \in [0, +\infty) \text{ and all } f \in \mathcal{L}^{p,q,n}.$$

Whenever this holds, the symbol $G(p, q; j, n, \vartheta)$ indicates its sharp constant.

3.2 Remarks. (i) The above inequality and its sharp constant are related to the space dimension d ; so we should write, say, $G_d(p, q; j, n, \vartheta)$ for the sharp constant. For the sake of simplicity, in the sequel d is fixed and omitted from most of our notations.

(ii) If (3.3) holds, for the related sharp constant we have the representation

$$G(p, q; j, n, \vartheta) = \sup_{f \in \mathcal{L}^{p,q,n} \setminus \{0\}} \frac{\|D^j f\|_{\mathcal{L}^r}}{\|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^{\vartheta}} . \quad (3.4)$$

(iii) The ratio in the right hand side of Eq. (3.4) is invariant under translations $f \mapsto f^a$ ($a \in \mathbb{R}^d$, see Eq. (2.25)). Thus, if f is a maximizer for (3.3), the same holds for each translated function f^a . \square

The ratio in (3.4) has another, less trivial invariance property described hereafter; this depends crucially on the definition (3.2) of r .

3.3 Proposition. *For p, q, j, n, ϑ, r as in (3.1) (3.2), let us consider the scaling transformation $f \mapsto f_\lambda$ (see Eq. (2.25)). For each $\lambda > 0$, the following holds:*

$$f \in \mathcal{L}^{p,q,n} \Rightarrow f_\lambda \in \mathcal{L}^{p,q,n} ; \quad (3.5)$$

$$\frac{\|D^j f_\lambda\|_{\mathcal{L}^r}}{\|f_\lambda\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f_\lambda\|_{\mathcal{L}^q}^{\vartheta}} = \frac{\|D^j f\|_{\mathcal{L}^r}}{\|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^{\vartheta}} \quad \text{for all } f \in \mathcal{L}^{p,q,n} \setminus \{0\} . \quad (3.6)$$

Therefore, if f is a maximizer for the Gagliardo-Nirenberg inequality (3.3), the same holds for each rescaled function f_λ .

Proof. To derive (3.6) use the scaling relations (2.28) (2.29), together with Eq. (3.2) for r . \square

The case $\vartheta = 1$ of the Gagliardo-Nirenberg inequality. Let $p, q \in [1, +\infty]$, $j, n \in [0, +\infty)$. We assume the last condition in (3.1) and Eq. (3.2) to be fulfilled with $\vartheta = 1$, so that

$$0 \leq \frac{1}{r} = \frac{1}{q} - \frac{n-j}{d} \leq 1 . \quad (3.7)$$

If one considers Definition 3.1 and applies it mechanically with $\vartheta = 1$, one obtains the following statement:

$$\mathcal{L}^{p,q,n} \subset \mathcal{L}^{p,r,j} , \quad (3.8)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|D^n f\|_{\mathcal{L}^q} \quad \text{for some } G \in [0, +\infty) \text{ and all } f \in \mathcal{L}^{p,q,n} .$$

The related sharp constant is indicated with $G(p, q; j, n)$.

It is natural to consider an extended inequality very similar to (3.8) but making no reference to \mathcal{L}^p , namely:

$$\dot{\mathcal{L}}^{q,n} \subset \dot{\mathcal{L}}^{r,j} , \quad (3.9)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|D^n f\|_{\mathcal{L}^q} \quad \text{for some } G \in [0, +\infty) \text{ and all } f \in \dot{\mathcal{L}}^{q,n} .$$

The corresponding sharp constant is indicated with $G(q; j, n)$.

3.4 Proposition. *Let $q \neq 1, +\infty$. The inequality (3.8) and the extended inequality (3.9) are equivalent; if they hold, $G(p, q; j, n) = G(q; j, n)$.*

Proof. An elementary density argument, reported for completeness in Appendix A. \square

3.2 The Sobolev inequality

This inequality depends on a set of parameters p, q, j, n, ϑ as in (3.1) and on an additional parameter t , with

$$1 \leq t \leq +\infty. \quad (3.10)$$

3.5 Definition. *Let $p, q, j, n, \vartheta, r, t$ be as in (3.1) (3.2) (3.10). The Sobolev inequality of order $(p, q; j, n, \vartheta|t)$ is the following statement:*

$$\mathcal{L}^{p,q,n} \subset \mathcal{L}^{p,r,j}, \quad (3.11)$$

$$\|D^j f\|_{\mathcal{L}^r} \leq S (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t} \quad \text{for some } S \in [0, +\infty) \text{ and all } f \in \mathcal{L}^{p,q,n}.$$

If this holds, the symbol $S(p, q; j, n, \vartheta|t)$ indicates the sharp constant.

3.6 Remarks. (i) Of course

$$S(p, q; j, n, \vartheta|t) = \sup_{f \in \mathcal{L}^{p,q,n} \setminus \{0\}} \frac{\|D^j f\|_{\mathcal{L}^r}}{(\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t}}. \quad (3.12)$$

(ii) The ratio in the right hand side of (3.12) and, consequently, the set of maximizers for (3.11) are invariant under translations $f \mapsto f^a$ ($a \in \mathbb{R}^d$, see Eq. (2.25)). \square

Scaling considerations. Let $p, q, j, n, \vartheta, r, t$ be as in (3.1) (3.2) (3.10). We consider the scaling transformation $f \mapsto f_\lambda$, see again Eq. (2.25), and its effect on the ratio in (3.12).

3.7 Lemma. *For each $\lambda > 0$, the following holds:*

$$\begin{aligned} & \frac{\|D^j f_\lambda\|_{\mathcal{L}^r}}{(\|f_\lambda\|_{\mathcal{L}^p}^t + \|D^n f_\lambda\|_{\mathcal{L}^q}^t)^{1/t}} \\ &= \frac{\|D^j f\|_{\mathcal{L}^r}}{[(\lambda^{-\vartheta(d/p-d/q+n)}\|f\|_{\mathcal{L}^p})^t + (\lambda^{(1-\vartheta)(d/p-d/q+n)}\|D^n f\|_{\mathcal{L}^q})^t]^{1/t}} \quad \text{for all } f \in \mathcal{L}^{p,q,n} \setminus \{0\} \\ & \text{(understanding both sides of (3.13) via (2.2), if } t = +\infty). \end{aligned} \quad (3.13)$$

Proof. Use again Eqs. (2.28) (2.29) and Eq. (3.2) for r . \square

3.3 Connecting the Gagliardo-Nirenberg and Sobolev inequalities

Let $\vartheta \in [0, 1]$, $t \in [1, +\infty]$ and $a, b \in [0, +\infty)$; it is well known that

$$a^{1-\vartheta}b^\vartheta \leq [(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t}(a^t+b^t)^{1/t}; \quad (3.14)$$

$$a^{1-\vartheta}b^\vartheta = [(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t}(a^t+b^t)^{1/t} \text{ if } \vartheta \neq 1 \text{ for } t < +\infty \text{ and } b = \left(\frac{\vartheta}{1-\vartheta}\right)^{1/t}a \quad (3.15)$$

(recall the conventions (2.1) (2.2); due to (2.1), here and in the sequel $\left(\frac{\vartheta}{1-\vartheta}\right)^0 = 1$ even for $\vartheta = 0, 1$). Let us also mention the following variant of (3.15), holding as well for $\vartheta = 1$ and $t < +\infty$: if $(a_\lambda), (b_\lambda)$ are nets with values in $(0, +\infty)$,

$$\lim_\lambda \frac{a_\lambda^{1-\vartheta}b_\lambda^\vartheta}{(a_\lambda^t+b_\lambda^t)^{1/t}} = [(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t} \quad \text{if } \lim_\lambda \frac{b_\lambda}{a_\lambda} = \left(\frac{\vartheta}{1-\vartheta}\right)^{1/t}. \quad (3.16)$$

The Gagliardo-Nirenberg and Sobolev inequalities can be connected using scaling considerations, combined with the above elementary facts; this was pointed out by several authors in special cases, see, e.g., [2] [11]. A general formulation of these ideas is as follows.

3.8 Proposition. *Let $p, q, j, n, \vartheta, r, t$ be as in (3.1) (3.2) (3.10); consider the Gagliardo-Nirenberg and Sobolev inequalities (3.3) and (3.11). Then, the following holds.*

(i) *If the Gagliardo-Nirenberg inequality holds, the Sobolev inequality holds as well and the corresponding sharp constants are related by*

$$S(p, q; j, n, \vartheta | t) \leq [(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t} G(p, q; j, n, \vartheta). \quad (3.17)$$

(ii) *In addition to (3.1) (3.2) (3.10), let*

$$\frac{d}{p} - \frac{d}{q} + n \neq 0; \quad (3.18)$$

then the Gagliardo-Nirenberg and Sobolev inequalities are equivalent. Whenever these inequalities hold, their sharp constants are related by

$$S(p, q; j, n, \vartheta | t) = [(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t} G(p, q; j, n, \vartheta). \quad (3.19)$$

(iii) *With the condition (3.18), assume the Gagliardo-Nirenberg inequality to hold and possess a maximizer f ; consider the rescaled functions f_λ ($\lambda > 0$). The function f_λ is a maximizer for the Sobolev inequality if we put*

$$\lambda := \left(\left(\frac{\vartheta}{1-\vartheta} \right)^{1/t} \frac{\|f\|_{\mathcal{L}^p}}{\|D^n f\|_{\mathcal{L}^q}} \right)^{1/(d/p-d/q+n)}. \quad (3.20)$$

In the cases $\vartheta = 0, 1$ and $t < +\infty$ (where (3.20) would give formally $\lambda = 0$ or $\lambda = +\infty$), the previous statement must be intended in this limit sense: the fundamental ratio (3.12) for the Sobolev inequality tends to the sharp constant if it is evaluated on f_λ and the limit $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$ is taken.

(iv) With the condition (3.18), assume the Sobolev inequality to hold and possess a maximizer f ; then f is as well a Gagliardo-Nirenberg maximizer.

Proof. We proceed in several steps, using for the sharp constants the shorthand notations

$$G(p, q; j, n, \vartheta) \equiv G, \quad S(p, q; j, n, \vartheta|t) \equiv S. \quad (3.21)$$

For the sake of brevity, in the sequel we assume

$$\vartheta \neq 0, 1, \quad t \neq +\infty; \quad (3.22)$$

the cases with $\vartheta = 0, 1$ and/or $t = +\infty$ are treated by simple variations of the considerations that follow (taking possibly the limits $\lambda \rightarrow 0^+$ or $\lambda \rightarrow +\infty$ mentioned in item (iii)).

Step 1. If the Gagliardo-Nirenberg inequality holds, the Sobolev inequality holds as well and its sharp constant fulfills

$$S \leq G[(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t} \quad (3.23)$$

(so, statement (i) of the proposition is proved). Let $f \in \mathcal{L}^{p,q,n}$. The inequality (3.3) with its sharp constant G states that $f \in \mathcal{L}^{p,r,j}$, and

$$\|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta; \quad (3.24)$$

on the other hand, Eq. (3.14) with $a = \|f\|_{\mathcal{L}^p}$ and $b = \|D^n f\|_{\mathcal{L}^q}$ gives

$$\|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta \leq [(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t} (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t}. \quad (3.25)$$

Due to (3.24) (3.25), the inequality (3.11) holds and its sharp constant S fulfills (3.23).

Step 2. With the condition (3.18), assume the Sobolev inequality to hold; then the Gagliardo-Nirenberg inequality holds as well, and its sharp constant fulfills

$$G \leq \frac{S}{[(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t}}. \quad (3.26)$$

To prove this, let us fix $f \in \mathcal{L}^{p,q,n}$; for each $\lambda > 0$, the Sobolev inequality for f_λ gives

$$\|D^j f_\lambda\|_{\mathcal{L}^r} \leq S (\|f_\lambda\|_{\mathcal{L}^p}^t + \|D^n f_\lambda\|_{\mathcal{L}^q}^t)^{1/t} \quad (3.27)$$

which implies, due to (3.13),

$$\|D^j f\|_{\mathcal{L}^r} \leq S (\lambda^{-\vartheta(d/p-d/q+n)t} \|f\|_{\mathcal{L}^p}^t + \lambda^{(1-\vartheta)(d/p-d/q+n)t} \|D^n f\|_{\mathcal{L}^q}^t)^{1/t} \equiv S F(\lambda). \quad (3.28)$$

By the arbitrariness of λ , this gives

$$\|D^j f\|_{\mathcal{L}^r} \leq S \inf_{\lambda \in (0, +\infty)} F(\lambda), \quad (3.29)$$

and one obtains by elementary means that

$$\inf_{\lambda \in (0, +\infty)} F(\lambda) = \frac{\|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta}{[(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t}} \quad (3.30)$$

(if $f \neq 0$, the inf of F is attained for λ as in (3.20)). Eqs. (3.29) (3.30) imply

$$\|D^j f\|_{\mathcal{L}^r} \leq \frac{S}{[(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t}} \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta; \quad (3.31)$$

this happens for all $f \in \mathcal{L}^{p,q,n}$, so the Gagliardo-Nirenberg inequality holds and its sharp constant G fulfills (3.26).

Step 3. Proof of statement (ii) in the text of the proposition. This follows immediately from Steps 1 and 2.

Step 4. With the condition (3.18), assume the Gagliardo-Nirenberg inequality to hold and admit a maximizer f ; then the rescaled function f_λ is a Sobolev maximizer, if λ is taken as in (3.20) (so, statement (iii) in the text of the proposition is proved). According to our assumptions,

$$f \in \mathcal{L}^{p,q,n} \setminus \{0\}, \quad \|D^j f\|_{\mathcal{L}^r} = G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta. \quad (3.32)$$

Since the ratio $\|D^j f\|_{\mathcal{L}^r} / (\|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta)$ is scale invariant (recall (3.6)), we also have

$$\|D^j f_\lambda\|_{\mathcal{L}^r} = G \|f_\lambda\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f_\lambda\|_{\mathcal{L}^q}^\vartheta \quad (3.33)$$

for any $\lambda > 0$. On the other hand, Eq. (3.15) with $a = \|f_\lambda\|_{\mathcal{L}^p}$ and $b = \|D^n f_\lambda\|_{\mathcal{L}^q}$ gives

$$\|f_\lambda\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f_\lambda\|_{\mathcal{L}^q}^\vartheta = [(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t} (\|f_\lambda\|_{\mathcal{L}^p}^t + \|D^n f_\lambda\|_{\mathcal{L}^q}^t)^{1/t} \quad (3.34)$$

under the condition

$$\|D^n f_\lambda\|_{\mathcal{L}^q} = \left(\frac{\vartheta}{1-\vartheta} \right)^{1/t} \|f_\lambda\|_{\mathcal{L}^p}; \quad (3.35)$$

due to the scaling properties (2.28) (2.29), this condition is equivalent to the equality

$$\lambda^{n-d/q} \|D^n f\|_{\mathcal{L}^q} = \left(\frac{\vartheta}{1-\vartheta} \right)^{1/t} \lambda^{-d/p} \|f\|_{\mathcal{L}^p}, \quad (3.36)$$

which is fulfilled if λ is chosen as in (3.20). With this choice for λ , Eqs. (3.33) (3.34) imply

$$\|D^j f_\lambda\|_{\mathcal{L}^r} = G [(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta]^{1/t} (\|f_\lambda\|_{\mathcal{L}^p}^t + \|D^n f_\lambda\|_{\mathcal{L}^q}^t)^{1/t}; \quad (3.37)$$

but $G[(1 - \vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t} = S$ due to (3.19), so f_λ is a Sobolev maximizer.

Step 5. With the condition (3.18), assume the Sobolev inequality to hold and admit a maximizer f ; then f is a Gagliardo-Nirenberg maximizer as well (so, statement (iv) in the text of the proposition is proved). According to our assumptions,

$$f \in \mathcal{L}^{p,q,n} \setminus \{0\} , \quad \|D^j f\|_{\mathcal{L}^r} = S (\|f\|_{\mathcal{L}^p}^t + \|D^n f\|_{\mathcal{L}^q}^t)^{1/t}; \quad (3.38)$$

from here and from the inequality (3.25) we infer

$$\|D^j f\|_{\mathcal{L}^r} \geq \frac{S}{[(1 - \vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t}} \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta . \quad (3.39)$$

But $S/[(1 - \vartheta)^{1-\vartheta}\vartheta^\vartheta]^{1/t} = G$ due to Eq. (3.19), so

$$\|D^j f\|_{\mathcal{L}^r} \geq G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta . \quad (3.40)$$

The reversed relation is just the Gagliardo-Nirenberg inequality, holding as well, so

$$\|D^j f\|_{\mathcal{L}^r} = G \|f\|_{\mathcal{L}^p}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^q}^\vartheta ; \quad (3.41)$$

this indicates that f is a Gagliardo-Nirenberg maximizer.

4 Validity conditions for the Gagliardo-Nirenberg and Sobolev inequalities

The original work of Nirenberg [27] proves the inequality (3.3) for

$$1 \leq p, q \leq +\infty , \quad 0 \leq \vartheta \leq 1 , \quad 0 \leq n, j < +\infty , \quad (4.1)$$

$$0 \leq \vartheta n - j \leq d \left(\frac{1 - \vartheta}{p} + \frac{\vartheta}{q} \right) , \quad \neg (1 < q < +\infty , \quad n = j + \frac{d}{q} , \quad \vartheta = 1) ,$$

with the additional assumption that j, n be *integer*, and defining $\|D^j f\|_{\mathcal{L}^r}$ (resp., $\|D^n f\|_{\mathcal{L}^q}$) in terms of the L^r (resp., L^q) norms of the partial derivatives of order j (resp., n) of f . In the above \neg indicates the logical negation (the case negated in (4.1) is called “exceptional” in [27]). We note that the conditions (4.1) imply $1 \leq r \leq +\infty$, where r is defined by (3.2).

To the best of our knowledge, the validity of the Gagliardo-Nirenberg inequality (3.3) with j, n possibly noninteger (and intending D^j, D^n , etc. as in the present paper) is nowadays known under conditions slightly more restrictive than (4.1), namely

$$1 < p, q < +\infty, \quad 0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad (4.2)$$

$$0 \leq \vartheta n - j < d \left(\frac{1 - \vartheta}{p} + \frac{\vartheta}{q} \right);$$

see Corollary 1.5 of [15] for a proof based on the representation of the homogeneous Sobolev spaces as special types of Triebel-Lizorkin spaces [13] [14] [30]. The conditions (4.2) imply $1 < r < +\infty$, with r as in (3.2).

Let us recall that, according to Proposition 3.8, for any $t \in [1, +\infty]$ the Sobolev inequality (3.11) is implied by (3.3), and the two inequalities are equivalent if $d/p - d/q + n \neq 0$.

In the next section we present *direct, autonomous* proofs for the validity of the inequalities (3.3) or (3.11) in all cases described by (4.1) with $p = q = 2$ (and $t = 2$), for both integer and noninteger values of j, n . Our analysis relies on a collection of methods not requiring the Triebel-Lizorkin formalism of [15]; these methods give the sharp constants in some subcases, and accurate bounds for them in the remaining ones. In all the subcases analyzed, we exhibit direct proofs and estimates on the related constants for either (3.3) or (3.11), according to convenience; next, we use Proposition 3.8 to infer conclusions on the other inequality.

5 Analysis of the \mathcal{L}^2 case

In this section we specialize the previous considerations to the case

$$p = q = t = 2; \quad (5.1)$$

so, our basic function space is \mathcal{L}^2 . In the sequel, for each $n \in [0, +\infty)$, the spaces $\dot{\mathcal{L}}^{2,n}$ and $\mathcal{L}^{2,2,n}$ defined via Eqs. (2.30)(2.32) are indicated with \dot{H}^n and H^n , respectively; thus

$$\dot{H}^n(\mathbb{R}^d) \equiv \dot{H}^n = \{f \in \Phi' \mid D^n f \in \mathcal{L}^2\}, \quad (5.2)$$

$$H^n(\mathbb{R}^d) \equiv H^n = \{f \in \mathcal{L}^2 \mid D^n f \in \mathcal{L}^2\}. \quad (5.3)$$

Using the fact that the Fourier transform \mathcal{F} maps isometrically \mathcal{L}^2 into itself, we readily infer the following for $f \in \Phi'$:

$$f \in \dot{H}^n \Leftrightarrow |\mathbf{k}|^n \mathcal{F}f \in \mathcal{L}^2; \quad f \in \dot{H}^n \Rightarrow \|D^n f\|_{\mathcal{L}^2} = \| |\mathbf{k}|^n \mathcal{F}f \|_{\mathcal{L}^2}; \quad (5.4)$$

$$f \in H^n \Leftrightarrow \mathcal{F}f, |\mathbf{k}|^n \mathcal{F}f \in \mathcal{L}^2 \Leftrightarrow \sqrt{1 + |\mathbf{k}|^{2n}} \mathcal{F}f \in \mathcal{L}^2; \quad (5.5)$$

$$f \in H^n \Rightarrow \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} = \|\sqrt{1 + |\mathbf{k}|^{2n}} \mathcal{F}f\|_{\mathcal{L}^2}.$$

One recognizes that H^n is the familiar \mathcal{L}^2 -based Sobolev space of Bessel potentials of order n [3] [24]. If n is integer, we can describe \dot{H}^n, H^n and $\|D^n f\|_{\mathcal{L}^2}$ in terms of the

partial derivatives $\partial_j : \Phi' \rightarrow \Phi'$ ($j = 1, \dots, d$). Indeed, by the Fourier representations of ∂_j and D^n we have the following for $f \in \Phi'$ and $n \in \mathbb{N}$:

$$f \in \dot{H}^n \Leftrightarrow \partial_{j_1 \dots j_n} f \in \mathcal{L}^2 \quad \text{for } j_1, \dots, j_n \in \{1, \dots, d\}; \quad (5.6)$$

$$\text{if } f \in \dot{H}^n, \quad \|D^n f\|_{\mathcal{L}^2}^2 = \sum_{j_1, \dots, j_n=1}^d \|\partial_{j_1 \dots j_n} f\|_{\mathcal{L}^2}^2; \quad (5.7)$$

$$\begin{aligned} f \in H^n &\Leftrightarrow f \in \mathcal{L}^2, \partial_{j_1 \dots j_n} f \in \mathcal{L}^2 \quad \text{for } j_1, \dots, j_n \in \{1, \dots, d\} \\ &\Leftrightarrow \partial_{j_1 \dots j_m} f \in \mathcal{L}^2 \quad \text{for } m \in \{0, \dots, n\} \text{ and } j_1, \dots, j_m \in \{1, \dots, d\} \end{aligned} \quad (5.8)$$

⁽¹¹⁾. The parameters of the \mathcal{L}^2 case are three real numbers j, n, ϑ ; the conditions (4.1) and the definition (3.2) for $r \equiv r(j, n, \vartheta)$ take the form

$$0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad 0 \leq \vartheta n - j \leq \frac{d}{2}, \quad \vartheta \neq 1 \text{ if } n = j + \frac{d}{2}; \quad (5.9)$$

$$\frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d}.$$

We note that the above conditions imply $0 \leq 1/r \leq 1/2$, i.e.,

$$2 \leq r \leq +\infty. \quad (5.10)$$

The Gagliardo-Nirenberg and Sobolev inequalities (3.3) (3.11) read

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad (5.11)$$

for some $G \in [0, +\infty)$ and all $f \in H^n$;

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad (5.12)$$

for some $S \in [0, +\infty)$ and all $f \in H^n$.

From now on, the sharp constants of these inequalities are denoted with $G(j, n, \vartheta)$ and $S(j, n, \vartheta)$, respectively.

As anticipated, in this section we present direct proofs of (5.11) and/or (5.12), fitted to the \mathcal{L}^2 framework, for all j, n, ϑ as in (5.9). Our analysis, carried over in the following subsections, follows the scheme (i)-(vii) already outlined in the Introduction.

¹¹To check some of the above statements, note the following:

- (i) $|\mathbf{k}|^{2n} |\mathcal{F}f|^2 = (\sum_{j=1}^d \mathbf{k}_j^2)^n |\mathcal{F}f|^2 = \sum_{j_1, \dots, j_n=1}^d |\mathbf{k}_{j_1} \dots \mathbf{k}_{j_n} \mathcal{F}f|^2$;
- (ii) if $m \in \{0, \dots, n\}$ and $j_1, \dots, j_m \in \{1, \dots, d\}$, $|\mathbf{k}_{j_1} \dots \mathbf{k}_{j_m} \mathcal{F}f|^2 \leq |\mathbf{k}|^{2m} |\mathcal{F}f|^2 \leq (1 + |\mathbf{k}|^{2m}) |\mathcal{F}f|^2 \leq C_{nm} (1 + |\mathbf{k}|^{2n}) |\mathcal{F}f|^2$ for a suitable, positive constant C_{nm} .

5.1 The elementary case $j = \theta n$

If we put $j = \vartheta n$ in Eq. (5.9) we obtain the conditions

$$0 \leq \vartheta \leq 1, \quad 0 \leq n < +\infty, \quad (5.13)$$

and the definition of r written therein gives $r = 2$. The Gagliardo-Nirenberg inequality (5.11) for this case reads

$$H^n \subset H^{\vartheta n}, \quad \|D^{\vartheta n} f\|_{\mathcal{L}^2} \leq \mathfrak{G} \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad (5.14)$$

for some $\mathfrak{G} \in [0, +\infty)$ and all $f \in H^n$.

As a matter of fact, this inequality is obtained by manipulations based on the Hölder inequality.

5.1 Proposition. *For $0 \leq \vartheta \leq 1$ and $0 \leq n < +\infty$, Eq. (5.14) holds with sharp constant $\mathfrak{G}(n, \vartheta) = 1$.*

Proof. It is divided in two steps, whose combination yields the thesis.

Step 1. Eq. (5.14) holds with sharp constant $\mathfrak{G}(n, \vartheta) \leq 1$. We use the Hölder inequality $\int uv \leq (\int u^p)^{1/p} (\int v^q)^{1/q}$ (for $u, v \geq 0$ and $p, q \geq 1$, $1/p + 1/q = 1$) and apply it with $u = |\mathcal{F}f|^{2(1-\vartheta)}$, $v = |\mathbf{k}|^{2\vartheta n} |\mathcal{F}f|^{2\vartheta}$, $p = 1/(1-\vartheta)$, $q = 1/\vartheta$; the result is the inequality $\| |\mathbf{k}|^{\vartheta n} \mathcal{F}f \|_{\mathcal{L}^2} \leq \| \mathcal{F}f \|_{\mathcal{L}^2}^{1-\vartheta} \| |\mathbf{k}|^n \mathcal{F}f \|_{\mathcal{L}^2}^{\vartheta}$ which is equivalent to

$$H^n \subset H^{\vartheta n}, \quad \|D^{\vartheta n} f\|_{\mathcal{L}^2} \leq \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad \text{for all } f \in H^n.$$

This gives the statement to be proved.

Step 2. The sharp constant of (5.14) is such that $\mathfrak{G}(n, \vartheta) \geq 1$. Of course, for each $f \in H^n \setminus \{0\}$ we have

$$\mathfrak{G}(n, \vartheta) \geq \frac{\|D^{\vartheta n} f\|_{\mathcal{L}^2}}{\|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta}}. \quad (5.15)$$

To go on, for $a \in \mathbb{R}^d$ and $\varepsilon \in (0, +\infty)$ let us introduce the function

$$\delta_{a\varepsilon} : \mathbb{R}^d \rightarrow [0, +\infty), \quad \delta_{a\varepsilon}(k) := \frac{\chi_{B_{a\varepsilon}}(k)}{v_\varepsilon}, \quad (5.16)$$

where $B_{a\varepsilon}$ is the ball in \mathbb{R}^d of center a and radius ε , $\chi_{B_{a\varepsilon}}$ is the characteristic function of this ball and $v_\varepsilon = \pi^{d/2} \varepsilon^d / \Gamma(d/2 + 1)$ is the volume of the ball, so that $\int_{\mathbb{R}^d} dk \delta_{a\varepsilon}(k) = 1$. For fixed a this function approaches the Dirac delta at a in the limit $\varepsilon \rightarrow 0$; more precisely, for each continuous function $g : \mathbb{R}^d \rightarrow \mathbb{C}$ we have

$$\int_{\mathbb{R}^d} dk g(k) \delta_{a\varepsilon} \rightarrow g(a) \quad \text{for } \varepsilon \rightarrow 0. \quad (5.17)$$

For a, ε as above, let us introduce the function

$$f_{a\varepsilon} := \mathcal{F}^{-1} \sqrt{\delta_{a\varepsilon}} . \quad (5.18)$$

Then $f_{a\varepsilon} \in \mathcal{L}^2$ and

$$\|f_{a\varepsilon}\|_{\mathcal{L}^2} = \|\mathcal{F}f_{a\varepsilon}\|_{\mathcal{L}^2} = \sqrt{\int_{\mathbb{R}^d} dk \delta_{a\varepsilon}(k)} = 1 ; \quad (5.19)$$

moreover, for each $m \in [0, +\infty)$ one has $D^m f_{a\varepsilon} \in \mathcal{L}^2$ and

$$\|D^m f_{a\varepsilon}\|_{\mathcal{L}^2} = \| |k|^m \mathcal{F}f_{a\varepsilon} \|_{\mathcal{L}^2} = \sqrt{\int_{\mathbb{R}^d} dk |k|^{2m} \delta_{a\varepsilon}(k)} \rightarrow |a|^m \quad \text{for } \varepsilon \rightarrow 0 . \quad (5.20)$$

To conclude, let us fix $a \in \mathbb{R}^d \setminus \{0\}$ and apply (5.15) with $f = f_{a\varepsilon}$; in this way we get

$$\mathfrak{S}(n, \vartheta) \geq \frac{\|D^{\vartheta n} f_{a\varepsilon}\|_{\mathcal{L}^2}}{\|f_{a\varepsilon}\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f_{a\varepsilon}\|_{\mathcal{L}^2}^{\vartheta}} \rightarrow \frac{|a|^{\vartheta n}}{(|a|^n)^{\vartheta}} = 1 \quad \text{for } \varepsilon \rightarrow 0 . \quad (5.21)$$

□

Let us pass to the Sobolev inequality (5.12), that in the present case $j = \vartheta n$ reads

$$H^n \subset H^{\vartheta n}, \quad \|D^{\vartheta n} f\|_{\mathcal{L}^2} \leq \mathfrak{S} \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad \text{for all } f \in H^n . \quad (5.22)$$

From the previous result on the Gagliardo-Nirenberg inequality we obtain the following result.

5.2 Corollary. *Let $0 \leq \vartheta \leq 1$, $0 < n < +\infty$. The inequality (5.22) holds with sharp constant*

$$\mathfrak{S}(n, \vartheta) = \sqrt{(1 - \vartheta)^{1-\vartheta} \vartheta^{\vartheta}} . \quad (5.23)$$

Proof. Use Proposition 5.1, together with Proposition 3.8 on the general relations between the Gagliardo-Nirenberg and Sobolev inequalities (especially, Eq. (3.19)).

□

5.3 Remark. Of course, Eq. (5.22) holds as well for $0 \leq \vartheta \leq 1$, $n = 0$ with sharp constant $\mathfrak{S}(0, \vartheta) = 1/\sqrt{2}$ (in fact, the inequality in (5.22) with $n = 0$ holds as an equality for $\mathfrak{S} = 1/\sqrt{2}$ and any $f \in \mathcal{L}^2$). □

5.2 The case $\vartheta = 1$

If we put $\vartheta = 1$ in the general conditions (5.9) we obtain

$$0 \leq j \leq n < j + \frac{d}{2}, \quad \frac{1}{r} = \frac{1}{2} - \frac{n-j}{d}; \quad (5.24)$$

note that $r \in [2, +\infty)$. The inequalities (5.11) (5.12) read

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{G} \|D^n f\|_{\mathcal{L}^2} \quad (5.25)$$

for some $\mathcal{G} \in [0, +\infty)$ and all $f \in H^n$;

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{S} \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad (5.26)$$

for some $\mathcal{S} \in [0, +\infty)$ and all $f \in H^n$.

In this case, it is natural to consider as well the extended Gagliardo-Nirenberg inequality, i.e., statement (3.9) with $p = q = 2$; this reads

$$\dot{H}^n \subset \dot{\mathcal{L}}^{r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{G} \|D^n f\|_{\mathcal{L}^2} \text{ for some } \mathcal{G} \in [0, +\infty) \text{ and all } f \in \dot{H}^n, \quad (5.27)$$

and is equivalent to (5.25) due to Proposition 3.4. The inequality (3.9) and, in particular, its \mathcal{L}^2 case (5.27) are strictly connected with the Hardy-Littlewood-Sobolev inequality [17] [32] concerning convolution with a power of the radius $|\mathbf{x}|$; this connection has a crucial role even in Sobolev's seminal paper [32] and is presented, e.g., by Mizohata [25] in a more up-to-date language. The sharp constants and maximizers of the Hardy-Littlewood-Sobolev inequality have been determined more recently by Lieb [19] for some cases, including the \mathcal{L}^2 case; by the previously mentioned equivalence, these results of Lieb can be used to determine the sharp constant and maximizers for (5.27), a fact somehow suggested by [19] and described more explicitly in [7].

The situation outlined above can be understood starting from the subcase $j = 0$ of (5.27); this is treated in the following proposition (and in the subsequent Remarks 5.5), very close to Theorem 1.1 of [7] ⁽¹²⁾.

5.4 Proposition. *Let*

$$0 \leq n < \frac{d}{2}, \quad \frac{1}{\hat{r}} = \frac{1}{2} - \frac{n}{d} \quad (5.28)$$

(where the second equation is understood as the definition of $\hat{r} \in [2, +\infty)$). Then

$$\dot{H}^n \subset \dot{\mathcal{L}}^{\hat{r}}, \quad \|f\|_{\dot{\mathcal{L}}^{\hat{r}}} \leq \mathcal{G} \|D^n f\|_{\mathcal{L}^2} \text{ for some } \mathcal{G} \in [0, +\infty) \text{ and all } f \in \dot{H}^n. \quad (5.29)$$

¹²See also the announcement of this theorem in [6]. The cited theorem of [6] [7] contains some imprecision, since it does not mention \dot{H}^n and always refers to H^n ; in particular, it seems to indicate that the maximizer f in Eq. (5.31) is in H^n for all n as in (5.28).

Moreover, the sharp constant in (5.29) is

$$\mathcal{G}(n) = \frac{1}{(4\pi)^{n/2}} \sqrt{\frac{\Gamma(d/2 - n)}{\Gamma(d/2 + n)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{n/d}. \quad (5.30)$$

A maximizer for (5.29) is

$$f := \frac{1}{(1 + |\mathbf{x}|^2)^{d/2-n}} = \frac{1}{2^{d/2-n-1}\Gamma(d/2 - n)} \mathcal{F}^{-1} \left(\frac{K_n(|\mathbf{k}|)}{|\mathbf{k}|^n} \right) \in \dot{H}^n, \quad (5.31)$$

where K_n denotes the modified Bessel function of the second kind (Macdonald function); note that $K_n(|\mathbf{k}|)/|\mathbf{k}|^n \in \mathcal{L}^1$. The above function f is in \mathcal{L}^2 (and thus in H^n) if and only if $n < d/4$.

Proof. It is divided in some steps; the main point is Step 1, reproducing a basic result of Lieb on the Hardy-Littlewood-Sobolev inequality. In the sequel we use the convolution $*$ and some of its properties, reviewed in Section 2.

Step 1 (a sharp Hardy-Littlewood-Sobolev inequality). Let n, \hat{r} be as in (5.28) and, in addition, $n \neq 0$. Then

$$h \in \mathcal{L}^2 \quad \Rightarrow \quad \frac{1}{|\mathbf{x}|^{d-n}} * h \in \mathcal{L}^{\hat{r}}, \quad \left\| \frac{1}{|\mathbf{x}|^{d-n}} * h \right\|_{\mathcal{L}^{\hat{r}}} \leq N_n \|h\|_{\mathcal{L}^2}, \quad (5.32)$$

$$N_n \equiv N_{nd} := \pi^{d/2-n/2} \frac{\Gamma(n/2)}{\Gamma(d/2 - n/2)} \sqrt{\frac{\Gamma(d/2 - n)}{\Gamma(d/2 + n)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{n/d}.$$

The inequality in (5.32) is fulfilled as an equality by the function

$$h := \frac{1}{|\mathbf{x}|^{d-n}} * \frac{1}{(1 + |\mathbf{x}|^2)^{d/2+n}} \in \mathcal{L}^2. \quad (5.33)$$

For all these statements see [19], Corollary 3.2, item (ii).

Step 2 (essentially, a reformulation of Step 1 via the fractional Laplacian). Let n, \hat{r} and $\mathcal{G}(n)$ be as in (5.28) (5.30). Then

$$h \in \mathcal{L}^2 \quad \Rightarrow \quad D^{-n}h \in \mathcal{L}^{\hat{r}}, \quad \|D^{-n}h\|_{\mathcal{L}^{\hat{r}}} \leq \mathcal{G}(n) \|h\|_{\mathcal{L}^2}, \quad (5.34)$$

with $\mathcal{G}(n)$ as in Eq. (5.30). The inequality in (5.34) is fulfilled as an equality by the function

$$h := D^{-n} \frac{1}{(1 + |\mathbf{x}|^2)^{d/2+n}} \in \mathcal{L}^2. \quad (5.35)$$

We first prove these statements for $n \neq 0$ (so that $0 < n < d/2$). To get the thesis, it suffices to write down the results of Step 1 and note that $|\mathbf{x}|^{-(d-n)} * \dots = Z_n^{-1} D^{-n}(\dots)$ with Z_n as in Eq. (2.24). In particular, the inequality $\| |\mathbf{x}|^{-(d-n)} * h \|_{\mathcal{L}^{\hat{r}}} \leq N_n \|h\|_{\mathcal{L}^2}$ of Step 1 becomes $\|D^{-n}h\|_{\mathcal{L}^{\hat{r}}} \leq Z_n N_n \|h\|_{\mathcal{L}^2}$, and one readily checks that $Z_n N_n = \mathcal{G}(n)$.

Let us pass to the case $n = 0$. Then all statements to be proved hold trivially since $\hat{r} = 2$ and $\mathcal{G}(0) = 1$; of course, the inequality (5.34) is fulfilled as an equality by any function in \mathcal{L}^2 , including the function in (5.34) with $n = 0$.

Step 3. Proof of all statements in the text of the proposition. Let again n , \hat{r} and $\mathcal{G}(n)$ be as in (5.28) and (5.30). By the very definition of \dot{H}^n , the map $f \mapsto D^n f$ is one-to-one between \dot{H}^n and \mathcal{L}^2 . Therefore, applying the results of Step 2 with $h = D^n f$ ($f \in \dot{H}^n$) we infer that

$$f \in \dot{H}^n \quad \Rightarrow \quad f \in \mathcal{L}^{\hat{r}}, \quad \|f\|_{\mathcal{L}^{\hat{r}}} \leq \mathcal{G}(n) \|D^n f\|_{\mathcal{L}^2} \quad (5.36)$$

and that the above inequality is fulfilled as an equality by the function

$$f = D^{-2n} \frac{1}{(1 + |\mathbf{x}|^2)^{d/2+n}} \in \dot{H}^n \quad (5.37)$$

(or by any multiple of it by a constant factor). Summing up, the extended Gagliardo-Nirenberg inequality holds in the case under consideration with $\mathcal{G}(n)$ as sharp constant and the function (5.37) as a maximizer.

Hereafter we show, via the related Fourier representations, that the maximizer (5.37) coincides, up to a constant factor, with the function in Eq.(5.31). Indeed, by definition $D^{-2n}(1 + |\mathbf{x}|^2)^{-(d/2+n)} = \mathcal{F}^{-1}(|\mathbf{k}|^{-2n} \mathcal{F}(1 + |\mathbf{x}|^2)^{-(d/2+n)})$; moreover, $\mathcal{F}(1 + |\mathbf{x}|^2)^{-(d/2+n)} = 2^{-(d/2+n-1)} \Gamma(d/2 + n)^{-1} |\mathbf{k}|^n K_{-n}(|\mathbf{k}|)$ by Lemma B.4 in Appendix B (with $\mu = \sigma = -n$) and $K_{-n} = K_n$, as well known. Thus

$$D^{-2n} \frac{1}{(1 + |\mathbf{x}|^2)^{d/2+n}} = \frac{1}{2^{d/2+n-1} \Gamma(d/2 + n)} \mathcal{F}^{-1} \left(\frac{K_n(|\mathbf{k}|)}{|\mathbf{k}|^n} \right); \quad (5.38)$$

on the other hand, using again Lemma B.4 (now with $\mu = \sigma = n$) we find that $K_n(|\mathbf{k}|)/|\mathbf{k}|^n \in \mathcal{L}^1$, and

$$\frac{1}{2^{d/2-n-1} \Gamma(d/2 - n)} \mathcal{F}^{-1} \left(\frac{K_n(|\mathbf{k}|)}{|\mathbf{k}|^n} \right) = \frac{1}{(1 + |\mathbf{x}|^2)^{d/2-n}}. \quad (5.39)$$

To conclude the proof, it remains to show that the maximizer $(1 + |\mathbf{x}|^2)^{-(d/2-n)}$ is in \mathcal{L}^2 (and thus in H^n) if and only if $n < d/4$; the verification is trivial. \square

5.5 Remarks. (i) For $n = 1$, the result of the above proposition was obtained by Aubin [4] and Talenti [34] some years before [19].

(ii) The analysis of Lieb on the Hardy-Littlewood-Sobolev inequality (5.32) shows as well that the function h in (5.33) is the *unique* maximizer up to translation, rescaling and multiplication by a constant factor. Therefore, one can make a similar statement for the maximizer (5.31) of (5.29); in the sequel we do not insist on such uniqueness issues. \square

Proposition 5.4 has a straightforward generalization to the case (5.24).

5.6 Corollary. *Let j, n, r be as in (5.24), and consider the extended Gagliardo-Nirenberg inequality (5.27):*

$$\dot{H}^n \subset \dot{\mathcal{L}}^{r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{G} \|D^n f\|_{\mathcal{L}^2} \quad \text{for some } \mathcal{G} \in [0, +\infty) \text{ and all } f \in \dot{H}^n.$$

This statement is true, and the sharp constant therein is

$$\mathcal{G}(j, n) = \frac{1}{(4\pi)^{(n-j)/2}} \sqrt{\frac{\Gamma(d/2 - n + j)}{\Gamma(d/2 + n - j)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{(n-j)/d}. \quad (5.40)$$

A maximizer for (5.27) is

$$f := D^{-j} \frac{1}{(1 + |\mathbf{x}|^2)^{d/2 - n + j}} \in \dot{H}^n; \quad (5.41)$$

this can be written as

$$f = \frac{1}{2^{d/2 - n + j - 1} \Gamma(d/2 - n + j)} \mathcal{F}^{-1} \left(\frac{K_{n-j}(|\mathbf{k}|)}{|\mathbf{k}|^n} \right) \quad (5.42)$$

(note that $K_{n-j}(|\mathbf{k}|)/|\mathbf{k}|^n$ makes sense in Ψ' as the product between $|\mathbf{k}|^{-j}$ and the \mathcal{L}^1 function $K_{n-j}(|\mathbf{k}|)/|\mathbf{k}|^{n-j}$).

One has $K_{n-j}(|\mathbf{k}|)/|\mathbf{k}|^n \in \mathcal{L}^1$ if and only if the stronger condition $n < j/2 + d/2$ holds; in this case $f \in \mathcal{L}^\infty$, and we have the representation

$$f = \frac{\Gamma(d/2 - j/2) \Gamma(d/2 - n + j/2)}{2^j \Gamma(d/2) \Gamma(d/2 - n + j)} {}_2F_1(d/2 - j/2, d/2 - n + j/2; d/2; -|\mathbf{x}|^2). \quad (5.43)$$

f is in \mathcal{L}^2 (and thus in H^n) if and only if the even stronger condition $n < j/2 + d/4$ holds.

Proof. Let $0 \leq s < d/2$, and write the inequality (5.29) with n replaced by s and with the sharp constant therein. This states that, with $1/\hat{r} = 1/2 - s/d$, $\dot{H}^s \subset \mathcal{L}^{\hat{r}}$ and $\|g\|_{\mathcal{L}^{\hat{r}}} \leq \mathcal{G}(s) \|D^s g\|_{\mathcal{L}^2}$ for all $g \in \dot{H}^s$; $\mathcal{G}(s)$ is as in (5.30) with n replaced by s , and the inequality holds as an equality if $g = 1/(1 + |\mathbf{x}|^2)^{d/2 - s}$.

Now let j, n, r be as in (5.24), and write the inequality $\|g\|_{\mathcal{L}^{\hat{r}}} \leq \mathcal{G}(s) \|D^s g\|_{\mathcal{L}^2}$ with $s := n - j$ and $g := D^j f$, $f \in \dot{H}^n$; this gives the inequality (5.25) with $\mathcal{G}(j, n) = \mathcal{G}(n - j)$, which has the explicit expression (5.40). Due to Proposition 5.4, (5.25) becomes an equality if we consider the element $f \in \dot{H}^n$ defined by Eq. (5.41), which is equivalent to

$$\mathcal{F}^{-1}(|\mathbf{k}|^j \mathcal{F} f) = \frac{1}{(1 + |\mathbf{x}|^2)^{d/2 - n + j}}; \quad (5.44)$$

but

$$\frac{1}{(1 + |\mathbf{x}|^2)^{d/2-n+j}} = \frac{1}{2^{d/2-n+j-1} \Gamma(d/2 - n + j)} \mathcal{F}^{-1} \left(\frac{K_{n-j}(|\mathbf{k}|)}{|\mathbf{k}|^{n-j}} \right) \quad (5.45)$$

and $K_{n-j}(|\mathbf{k}|)/|\mathbf{k}|^{n-j} \in \mathcal{L}^1$; to prove these statements, use Lemma B.4 in Appendix B with $\mu = \sigma = n - j$. In view of (5.45), Eq. (5.44) is equivalent to Eq. (5.42)

$$f = \frac{1}{2^{d/2-n+j-1} \Gamma(d/2 - n + j)} \mathcal{F}^{-1} \left(\frac{K_{n-j}(|\mathbf{k}|)}{|\mathbf{k}|^n} \right).$$

Now, using Lemma B.4 with $\mu = n - j$ and $\sigma = n$ we obtain the remaining statements to be proved: $K_{n-j}(|\mathbf{k}|)/|\mathbf{k}|^n \in \mathcal{L}^1$ if and only if $n < j/2 + d/2$, in this case $f \in \mathcal{L}^\infty$ and Eq. (5.43) holds; $f \in \mathcal{L}^2$ if and only if $n < j/2 + d/4$. \square

5.7 Remark. Obviously enough, one would like to generalize Eq. (5.43) to all j, n as in (5.24), removing the limitation $n < j/2 + d/2$. To illustrate the related difficulties, it suffices to consider the right hand side of (5.43) for fixed j and $n \rightarrow (j/2 + d/2)^-$. In this limit, the term $\Gamma(d/2 - n + j/2)$ diverges and the hypergeometric function in (5.43) becomes formally ${}_2F_1(d/2 - j/2, 0; d/2; -|\mathbf{x}|^2) = 1$. However, a constant function represents the zero element of Φ' ⁽¹³⁾; so, the right hand side of Eq. (5.43) gives, in the limit $n \rightarrow (j/2 + d/2)^-$, an indeterminate form in Φ' . We leave to future work the analysis of this problem and the discussion of (5.43) for $n > j/2 + d/2$ (perhaps possible by analytic continuation arguments). \square

Here is another consequence of the previous results.

5.8 Corollary. *Let j, n, r be as in (5.24), and consider the inequalities (5.11) (5.12) :*

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{G} \|D^n f\|_{\mathcal{L}^2} \quad \text{for some } \mathcal{G} \in [0, +\infty) \text{ and all } f \in H^n;$$

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{S} \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad \text{for some } \mathcal{S} \in [0, +\infty) \text{ and all } f \in H^n.$$

These are true. Their sharp constants $\mathcal{G}(j, n)$, $\mathcal{S}(j, n)$ are both equal to the right hand side of Eq. (5.40), if $n \neq 0$; in the subcase $n = 0$, implying $j = 0$, the sharp constants are $\mathcal{G}(0, 0) = 1$ and $\mathcal{S}(0, 0) = 1/\sqrt{2}$.

¹³Let us recall that Φ' can be identified with \mathcal{S}' modulo the polynomial functions: see the footnote before Eq. (2.8).

Proof. Due to Proposition 3.4, the inequality (5.11) is equivalent to the extended inequality (5.27) of Corollary 5.6; moreover they have the same sharp constant $\mathcal{G}(j, n)$, given by (5.40). On the other hand Proposition 3.8 with $p = q = t = 2$ and $\vartheta = 1$ ensures that, for $n \neq 0$, the inequalities (5.11) (5.12) are equivalent and possess the same sharp constant. The statements on the subcase $n = 0$ (and $j = 0$) are obvious (and make reference to already mentioned facts, see Remark 5.3). \square

5.3 An “almost general” \mathcal{L}^2 case: proof of the Gagliardo-Nirenberg and Sobolev inequalities and upper bounds for their sharp constants

In this subsection we make the assumptions

$$0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad 0 \leq \vartheta n - j < \frac{d}{2}; \quad \frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d}, \quad (5.46)$$

which differ from the general \mathcal{L}^2 conditions (5.9) since they exclude the case with $\vartheta n - j = d/2$ (occurring if only if $n \neq 0$ and $\vartheta = j/n + d/2n$); in this sense, we are considering an almost general \mathcal{L}^2 case. We note that $r \in [2, +\infty)$.

We consider the inequalities (5.11) (5.12) and their sharp constants $G(j, n, \vartheta)$, $S(j, n, \vartheta)$. Following the approach of [16] (see Corollary 2.3), we show how to infer (5.11) and an upper bound on $G(j, n, \vartheta)$ using results on the cases $j = \vartheta n$ and $\vartheta = 1$ (see our subsections 5.1 and 5.2) ⁽¹⁴⁾; this has implications on the Sobolev inequality, according to Proposition 3.8.

5.9 Proposition. *Let j, n, ϑ, r be as in (5.46); consider the Gagliardo-Nirenberg inequality (5.11)*

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad \text{for all } f \in H^n.$$

This statement is true, and the related sharp constant $G(j, n, \vartheta)$ has the upper bound

$$G(j, n, \vartheta) \leq G_+(j, n, \vartheta), \quad (5.47)$$

$$G_+(j, n, \vartheta) := \frac{1}{(4\pi)^{(\vartheta n - j)/2}} \sqrt{\frac{\Gamma(d/2 - \vartheta n + j)}{\Gamma(d/2 + \vartheta n - j)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{(\vartheta n - j)/d}.$$

The equality $G(j, n, \vartheta) = G_+(j, n, \vartheta)$ holds in the following subcases:

- (a) $j = \vartheta n$, where $G(j, n, \vartheta) = 1$ due to Proposition 5.1;
- (b) $\vartheta = 1$, where $G(j, n, \vartheta) = \mathcal{G}(j, n)$ as in (5.40).

¹⁴To be precise, Corollary 2.3 of [16] is about the Gagliardo-Nirenberg inequality (3.3) with p, q arbitrary and $j = 0$, whereas here $p = q = 2$ and j can be nonzero.

Proof. The validity of Eq. (5.11) and the sharp constants in the subcases (a)(b) are known from subsections 5.1, 5.2; the equality $G(j, n, \vartheta) = G_+(j, n, \vartheta)$ in these subcases is checked immediately comparing the known values of the sharp constants with Eq. (5.47).

In the rest of the proof we assume $\vartheta \neq 0, 1$. With this assumption (and with the previous ones in (5.46) for j, n, ϑ, r), we see that Eq. (5.24) holds with n replaced by ϑn and j, r as before; therefore Corollary 5.8 gives

$$H^{\vartheta n} \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq \mathcal{G}(j, \vartheta n) \|D^{\vartheta n} f\|_{\mathcal{L}^2} \quad \text{for all } f \in H^{\vartheta n}, \quad (5.48)$$

with $\mathcal{G}(j, \vartheta n)$ defined following Eq. (5.40); one checks that

$$\mathcal{G}(j, \vartheta n) = G_+(j, n, \vartheta) \quad \text{as in (5.47)}. \quad (5.49)$$

On the other hand, due to Proposition 5.1 we have

$$H^n \subset H^{\vartheta n}, \quad \|D^{\vartheta n} f\|_{\mathcal{L}^2} \leq \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad \text{for all } f \in H^n. \quad (5.50)$$

Eqs. (5.48) (5.49) (5.50) give

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G_+(j, n, \vartheta) \|f\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n f\|_{\mathcal{L}^2}^{\vartheta} \quad \text{for all } f \in H^n; \quad (5.51)$$

summing up, (5.11) holds and $G(j, n, \vartheta)$ is bounded from above by $G_+(j, n, \vartheta)$. \square

5.10 Corollary. *Let j, n, ϑ, r be as in (5.46); consider the Sobolev inequality (5.12)*

$$\|D^j f\|_{\mathcal{L}^r} \leq S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad \text{for all } f \in H^n.$$

This statement is true, and the related sharp constant $S(j, n, \vartheta)$ has the upper bound

$$S(j, n, \vartheta) \leq S_+(j, n, \vartheta), \quad (5.52)$$

$$S_+(j, n, \vartheta) := \sqrt{(1-\vartheta)^{1-\vartheta} \vartheta^{\vartheta}} G_+(j, n, \vartheta), \quad G_+(j, n, \vartheta) \text{ as in (5.47)}.$$

The equality $S(j, n, \vartheta) = S_+(j, n, \vartheta)$ holds in the following cases:

- (a) $n > 0$, $j = \vartheta n$, where $S(j, n, \vartheta) = \sqrt{(1-\vartheta)^{1-\vartheta} \vartheta^{\vartheta}}$ due to Corollary 5.2;
- (b) $n > 0$, $\vartheta = 1$, where $S(j, n, \vartheta) = \mathcal{G}(j, n)$ as in (5.40) due to Corollary 5.8.

Proof. Use Propositions 5.9 and 3.8. \square

Let us remark that, due to one of the Gamma function terms in (5.47), the upper bound $G_+(j, n, \vartheta)$ diverges if we fix j, n and consider the limit $\vartheta \rightarrow j/n + d/2n$; the same can be said of the bound $S_+(j, n, \vartheta)$ defined by (5.52). The case $\vartheta = j/n + d/2n$, which is excluded from the conditions (5.46), is attacked with a different strategy in the next subsection, where we even obtain the sharp constants.

5.4 The \mathcal{L}^∞ subcase

In this subsection we assume

$$0 < n < +\infty, \quad 0 \leq j < +\infty, \quad \vartheta \equiv \vartheta(j, n) := \frac{j}{n} + \frac{d}{2n} < 1 \quad (5.53)$$

and note that, with this choice of ϑ , the general definition of r in Eq. (5.9) gives

$$r = +\infty. \quad (5.54)$$

Our subsequent consideration will frequently refer to the space $\mathcal{L}^\infty = C_0(\mathbb{R}^d, \mathbb{C})$, see Eq. (2.12). The inequalities (5.11) (5.12) read

$$H^n \subset \mathcal{L}^{2,\infty,j}, \quad \|D^j f\|_{\mathcal{L}^\infty} \leq G \|f\|_{\mathcal{L}^2}^{1-\vartheta(j,n)} \|D^n f\|_{\mathcal{L}^2}^{\vartheta(j,n)} \quad (5.55)$$

for some G and all $f \in H^n$,

$$H^n \subset \mathcal{L}^{2,\infty,j}, \quad \|D^j f\|_{\mathcal{L}^\infty} \leq S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} \quad (5.56)$$

for some S and all $f \in H^n$.

Differently from the previous subsections, here we first give a result for (5.56) and then present its implications for (5.55); this approach gives the sharp constants for both inequalities.

In item (iv) of the Introduction we have already mentioned that the inequalities (5.55) (5.56) have been analyzed for $j = 0$ by Ilyin [18], who determined the sharp constants and the maximizers. This author is mainly interested in Eq. (5.55), so he does not present a specific statement about (5.56); however, he essentially derives Eq. (5.56) with its sharp constant and uses this result as an intermediate step towards Eq. (5.55) ⁽¹⁵⁾. In this paragraph we generalize the results of Ilyin to the case of arbitrary j ; after deriving the sharp constant and a maximizer for Eq. (5.56), we use Proposition 3.8 to obtain the analogous results on (5.55).

Let us repeat another fact anticipated in item (iv) of the Introduction. The maximizer for (5.55) (5.56) that we present coincides for $j = 0$ with the one determined by Ilyin ⁽¹⁶⁾; besides giving its Fourier representation (as in [18] for $j = 0$), we also express it in terms of the space variables using the Fox H -function or the Meijer G -function (for arbitrary j). The definitions of the above mentioned special functions are summarized in a specific subsection of Appendix B, where we also give some basic references about them. We remark that the G -function is a special case

¹⁵For completeness, let us mention that Eq. (5.56) for $j = 0$ has a structure very similar to the inequality $\|f\|_{\mathcal{L}^\infty} \leq K \|\sqrt{1 + D^2}^n f\|_{\mathcal{L}^2}^2$, for which we have given the sharp constant and a maximizer in our previous work [26].

¹⁶This is not at all surprising, since Ilyin proves uniqueness of the maximizer for (5.55) with $j = 0$ up to translation, rescaling and multiplication by a constant.

of the H -function, more frequently implemented in standard packages for symbolic or numerical computations with special functions; due to this fact, we emphasize the use of G whenever possible. For certain choices of the parameters, the H - or G -functions considered hereafter are in fact elementary functions; some examples are given in Section 6.

After all these preliminaries, we focus the attention on (5.56).

5.11 Proposition. *For j, n and $\vartheta(j, n)$ as in (5.53), we have the following.*

(i) *The Sobolev inequality (5.56) holds and its sharp constant is*

$$S(j, n) = \frac{1}{2^{d/2} \pi^{d/4-1/2} \sqrt{\Gamma(d/2) n \sin(\pi \vartheta(j, n))}}. \quad (5.57)$$

A maximizer for (5.56) is the function

$$f := \mathcal{F}^{-1} \left(\frac{|\mathbf{k}|^j}{1 + |\mathbf{k}|^{2n}} \right). \quad (5.58)$$

The above f , being the inverse Fourier transform of an \mathcal{L}^1 function, is in \mathcal{L}^∞ ; moreover, f can be expressed as follows using the Bessel function $J_{d/2-1}$:

$$f = F_{jn}(|\mathbf{x}|), \quad F_{jn} \in C([0, +\infty), \mathbb{R}), \quad (5.59)$$

$$F_{jn}(\rho) := \int_0^{+\infty} d\xi \frac{J_{d/2-1}(\rho \xi)}{(\rho \xi)^{d/2-1}} \frac{\xi^{d+j-1}}{1 + \xi^{2n}} \text{ for } \rho > 0, \quad F_{jn}(0) = \frac{\pi}{2^{d/2} \Gamma(d/2) n \sin(\pi \frac{j+d}{2n})}.$$

(ii) *The function F_{jn} of Eq. (5.59) can be expressed as a Fox H -function, in the following way: for all $\rho \in [0, +\infty)$,*

$$F_{jn}(\rho) = \frac{1}{2^{d/2} n} H \left(\begin{matrix} (1 - \frac{j+d}{2n}, \frac{1}{n}); \\ (0, 1), (1 - \frac{j+d}{2n}, \frac{1}{n}); (1 - \frac{d}{2}, 1) \end{matrix} \middle| \left(\frac{\rho}{2} \right)^2 \right). \quad (5.60)$$

F_{jn} can also be expressed in terms of the Meijer G -function if n is rational. More precisely, if

$$n = \frac{N}{M} \quad N, M \in \{1, 2, 3, \dots\} \quad (5.61)$$

we have the following:

$$F_{jn}(\rho) = \frac{M}{2^{d/2+M-1} \pi^{M-1} N^{d/2}} G \left(\begin{matrix} a_1, \dots, a_N; \\ b_1, \dots, b_{N+M}; b_1^*, \dots, b_N^* \end{matrix} \middle| \left(\frac{\rho}{2N} \right)^{2N} \right), \quad (5.62)$$

where the parameters labeling the G -function are defined as follows:

$$a_\ell := 1 - \frac{j+d}{2N} - \frac{\ell-1}{M} \quad \text{for } \ell = 1, \dots, M ; \quad (5.63)$$

$$b_h := \frac{h-1}{N} \quad \text{for } h = 1, \dots, N , \quad b_{N+h} := -\frac{j+d}{2N} + \frac{h}{M} \quad \text{for } h = 1, \dots, M ;$$

$$b_\ell^* := 1 - \frac{d}{2N} - \frac{\ell-1}{N} \quad \text{for } \ell = 1, \dots, N .$$

Proof. In the sequel we frequently refer to the integral

$$I(j, n) := \int_{\mathbb{R}^d} dk \frac{|k|^{2j}}{1 + |k|^{2n}} , \quad (5.64)$$

which is finite due to the assumptions on j, n in (5.53) and given by

$$I(j, n) = \frac{\pi^{d/2+1}}{\Gamma(d/2)n \sin(\pi\vartheta(j, n))} \quad (5.65)$$

(see Lemma B.1 in Appendix B ⁽¹⁷⁾). Our argument is divided in two steps.

Step 1. The Sobolev inequality (5.56) holds and its sharp constant $S(j, n)$ fulfills

$$S(j, n) \leq \frac{\sqrt{I(j, n)}}{(2\pi)^{d/2}} . \quad (5.66)$$

To prove this we note that

$$g \in \mathcal{L}^1 \Rightarrow \mathcal{F}^{-1}g \in \mathcal{L}^\infty, \quad \|\mathcal{F}^{-1}g\|_{\mathcal{L}^\infty} \leq \frac{1}{(2\pi)^{d/2}} \|g\|_{\mathcal{L}^1} ; \quad (5.67)$$

this is the special case $p = 1$ of the Hausdorff-Young inequality (2.19) (which is in fact derived by elementary means, since the relation $\mathcal{F}^{-1}g(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} dk e^{ikx} g(k)$ implies $|\mathcal{F}^{-1}g(x)| \leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} dk |g(k)|$ for all x).

Let us fix $f \in H^n$. Writing $D^j f = \mathcal{F}^{-1}(|\mathbf{k}|^j \mathcal{F}f)$ and using (5.67) with $g = |\mathbf{k}|^j \mathcal{F}f$ we obtain

$$D^j f \in \mathcal{L}^\infty, \quad \|D^j f\|_{\mathcal{L}^\infty} \leq \frac{1}{(2\pi)^{d/2}} \| |\mathbf{k}|^j \mathcal{F}f \|_{\mathcal{L}^1} , \quad (5.68)$$

provided that $|\mathbf{k}|^j \mathcal{F}f \in \mathcal{L}^1$. In order to check this statement, we write

$$|\mathbf{k}|^j \mathcal{F}f = \frac{|\mathbf{k}|^j}{\sqrt{1 + |\mathbf{k}|^{2n}}} \sqrt{1 + |\mathbf{k}|^{2n}} \mathcal{F}f$$

and use the Hölder inequality; this ensures that $|\mathbf{k}|^j \mathcal{F}f$ is actually in \mathcal{L}^1 , with

¹⁷Use item (ii) of this lemma with $a = j + d/2$ and $b = n$, so that (by (5.53)) $b > a > 0$ and $a/b = \vartheta(j, n)$.

$$\begin{aligned}
\| |\mathbf{k}|^j \mathcal{F}f \|_{\mathcal{L}^1} &\leq \left\| \frac{|\mathbf{k}|^j}{\sqrt{1+|\mathbf{k}|^{2n}}} \right\|_{\mathcal{L}^2} \| \sqrt{1+|\mathbf{k}|^{2n}} \mathcal{F}f \|_{\mathcal{L}^2} \\
&= \sqrt{I(j,n)} \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}
\end{aligned} \tag{5.69}$$

(as for the last equality, see (5.5) and (5.64)). Inserting (5.69) into (5.68), we conclude

$$D^j f \in \mathcal{L}^\infty, \quad \|D^j f\|_{\mathcal{L}^\infty} \leq \frac{\sqrt{I(j,n)}}{(2\pi)^{d/2}} \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}. \tag{5.70}$$

This proves the inequality (5.56) and gives the bound (5.66) on its sharp constant.
Step 2. Let $f := \mathcal{F}^{-1} \left(\frac{|\mathbf{k}|^j}{1+|\mathbf{k}|^{2n}} \right)$, as in (5.58). Then f , being the Fourier transforms of an \mathcal{L}^1 function, is in \mathcal{L}^∞ ; f possesses the features described by Eqs. (5.59)-(5.63), and is in H^n . Moreover,

$$S(j,n) \geq \frac{\|D^j f\|_{\mathcal{L}^\infty}}{\sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}} = \frac{\sqrt{I(j,n)}}{(2\pi)^{d/2}}. \tag{5.71}$$

Indeed, f fits to the framework of Lemma B.3 in Appendix B⁽¹⁸⁾. Being the Fourier transform of an \mathcal{L}^1 function, f is in \mathcal{L}^∞ and can be described via Eqs. (5.59)-(5.63) due to the cited lemma. To go on, we note that Eq. (5.5) and the convergence of the integral $I(j,n)$ in (5.64) yield the following statements

$$\sqrt{1+|\mathbf{k}|^{2n}} \mathcal{F}f = \frac{|\mathbf{k}|^j}{\sqrt{1+|\mathbf{k}|^{2n}}} \in \mathcal{L}^2, \quad \text{whence } f \in H^n; \tag{5.72}$$

$$\sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2} = \|\sqrt{1+|\mathbf{k}|^{2n}} \mathcal{F}f\|_{\mathcal{L}^2} = \sqrt{I(j,n)}. \tag{5.73}$$

Let us pass to

$$D^j f = \mathcal{F}^{-1} \left(\frac{|\mathbf{k}|^{2j}}{1+|\mathbf{k}|^{2n}} \right) \tag{5.74}$$

(automatically in \mathcal{L}^∞ , due to (5.56)). This function also fits to Lemma B.3 which ensures, amongst else,

$$\|D^j f\|_{\mathcal{L}^\infty} = D^j f(0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk \frac{|k|^{2j}}{1+|k|^{2n}} = \frac{I(j,n)}{(2\pi)^{d/2}}. \tag{5.75}$$

Eqs. (5.73) (5.75) imply

$$\frac{\|D^j f\|_{\mathcal{L}^\infty}}{\sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}} = \frac{\sqrt{I(j,n)}}{(2\pi)^{d/2}};$$

¹⁸This lemma must be used with $a = j/2 + d/2$ and $b = n$. It is $a, b > 0$ and $b > a$, since $a/b = \vartheta(j,n) - j/2n < \vartheta(j,n) < 1$.

this ratio is obviously bounded from above by $S(j, n)$, so we have the thesis (5.71).
Conclusion of the proof. Steps 1 and 2 indicate that

$$S(j, n) = \frac{\sqrt{I(j, n)}}{(2\pi)^{d/2}} = \frac{\|D^j f\|_{\mathcal{L}^\infty}}{\sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}}, \quad (5.76)$$

with f as in Eqs. (5.58)–(5.59). Thus f is a maximizer; finally, expressing $I(j, n)$ via (5.65) we obtain Eq. (5.57) for $S(j, n)$. \square

5.12 Corollary. *For j, n and $\vartheta(j, n) \equiv \vartheta$ as in (5.53), the Gagliardo-Nirenberg inequality (5.55) holds and its sharp constant $G(j, n)$ is given by*

$$G(j, n) = \frac{S(j, n)}{\sqrt{(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta}} = \frac{1}{2^{d/2}\pi^{d/4-1/2}\sqrt{\Gamma(d/2)(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta}n\sin(\pi\vartheta)}. \quad (5.77)$$

The function f defined by Eq. (5.58) is a maximizer for (5.55).

Proof. Use Propositions 5.11 and 3.8. \square

5.13 Remark. From (5.57)–(5.77) it is evident that $S(j, n) = S(j', n)$ and $G(j, n) = G(j', n)$ if $(j, n), (j', n)$ fulfill conditions (5.53) and $\vartheta(j, n) + \vartheta(j', n) = 1$. \square

5.5 Another “almost general” \mathcal{L}^2 case: alternative upper bounds for the Gagliardo-Nirenberg and Sobolev sharp constants

In this subsection we make the assumptions

$$0 \leq \vartheta < 1, \quad 0 \leq n, j < +\infty, \quad 0 \leq \vartheta n - j \leq \frac{d}{2} \quad ; \quad \frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d}, \quad (5.78)$$

which differ from the general \mathcal{L}^2 conditions (5.9) since they exclude the case $\vartheta = 1$; in this sense we are considering an almost general \mathcal{L}^2 case, slightly different from the one of subsection 5.3.

Hereafter we use the strategy of subsection 5.4, i.e., we first derive a result on the Sobolev inequality and then we point out its implications for the Gagliardo-Nirenberg inequality; this approach produces upper bounds for the sharp constants of both inequalities. Let us note that the conditions (5.78) differ from the conditions (5.46) of subsection 5.3 only at boundary values. In the intersection of (5.46) with (5.78), we have two alternative upper bounds for the Gagliardo-Nirenberg and Sobolev sharp constants; a unified view of both types of upper bounds is proposed in subsection 5.6.

5.14 Proposition. Let j, n, ϑ, r be as in (5.78); consider the Sobolev inequality (5.12)

$$H^n \subset \mathcal{L}^{2,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq S \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}.$$

This is true and its sharp constant $S(j, n, \vartheta)$ has the upper bound

$$S(j, n, \vartheta) \leq S_{++}(j, n, \vartheta), \quad S_{++}(j, n, \vartheta) := \frac{E(j, n, \vartheta) F(j, n, \vartheta)}{\pi^{\vartheta n/2 - j/2}}; \quad (5.79)$$

here

$$E(j, n, \vartheta) := \frac{(1 + 2j/d - 2\vartheta n/d)^{d/4 + j/2 - \vartheta n/2}}{(1 - 2j/d + 2\vartheta n/d)^{d/4 - j/2 + \vartheta n/2}}; \quad (5.80)$$

$$F(j, n, \vartheta) := \left(\frac{\Gamma\left(\frac{d(1-\vartheta)}{2(\vartheta n - j)}\right) \Gamma\left(\frac{d\vartheta}{2(\vartheta n - j)}\right)}{n \Gamma\left(\frac{d}{2}\right) \Gamma\left(\frac{d}{2(\vartheta n - j)}\right)} \right)^{\frac{\vartheta n - j}{d}} \quad \text{if } j < \vartheta n, \quad (5.81)$$

$$F(\vartheta n, n, \vartheta) := \sqrt{(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta}$$

(note that, if $n \neq 0$, $F(\vartheta n, n, \vartheta) = \lim_{j \rightarrow (\vartheta n)^-} F(j, n, \vartheta)$).

The equality $S_{++}(j, n, \vartheta) = S(j, n, \vartheta)$ holds in the following subcases:

- (a) $\vartheta = j/n$ with $n \neq 0$, where $S(j, n, \vartheta) = \sqrt{(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta}$ due to Corollary 5.2;
- (b) $\vartheta = j/n + d/2n$ with $n \neq 0$, where $S(j, n, \vartheta) =$ right hand side of Eq. (5.57).

Before proving the above statements, we would like to point out their connections with earlier works. In our previous paper [26] we have considered the inequality $\|f\|_{\mathcal{L}^r} \leq K \|\sqrt{1 + D^2}^n f\|_{\mathcal{L}^2}$, very similar to the case $j = 0$ of (5.12), and we have determined upper (and lower) bounds for the sharp constant $K \equiv K(n, r)$. Hereafter we adapt the arguments of [26], replacing the norm $\|\sqrt{1 + D^2}^n f\|_{\mathcal{L}^2}$ employed therein with the equivalent norm $\sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}$. The difference between these norms has some implications from the computational viewpoint; apart from this (and from the extension to the case $j \neq 0$), the forthcoming proof of Proposition 5.14 follows the main ideas of [26], e.g., a combination of the Hölder and Hausdorff-Young inequalities to derive the upper bound in Eq. (5.79). Let us also mention that, for $d = 1$ and $j = 0$, the upper bound of (5.79) has been (announced in [6] and) derived in [7] along similar lines.

Proof of Proposition 5.14. For $n = 0$, all statements to be proved are trivial; in fact, in this case we have $j = 0$, $r = 2$, $S_{++}(0, 0, \vartheta) = \sqrt{(1 - \vartheta)^{1-\vartheta} \vartheta^\vartheta} \geq 1/\sqrt{2}$ and $1/\sqrt{2}$ is just the sharp constant of the Sobolev inequality, see Remark 5.3.

From here to the end of the proof we assume $n \neq 0$: our argument is divided in several steps.

Step 1. Defining r', s . In the sequel we denote with r', s the solutions of the equations

$$\frac{1}{r} + \frac{1}{r'} = 1, \quad \frac{1}{r'} = \frac{1}{2} + \frac{1}{s}. \quad (5.82)$$

Recalling that $r \in [2, +\infty]$, see (5.10), we infer from (5.82) that

$$r' \in [1, 2] , \quad s \in [2, +\infty] . \quad (5.83)$$

From the explicit expression of r in (5.78) we readily obtain

$$r' = \frac{2d}{d + 2(\vartheta n - j)} , \quad s = \frac{d}{\vartheta n - j} . \quad (5.84)$$

Step 2. One has

$$\frac{|\mathbf{k}|^j}{\sqrt{1 + |\mathbf{k}|^{2n}}} \in \mathcal{L}^s , \quad \left\| \frac{|\mathbf{k}|^j}{\sqrt{1 + |\mathbf{k}|^{2n}}} \right\|_{\mathcal{L}^s} = \pi^{\vartheta n/2 - j/2} F(j, n, \vartheta) \quad (5.85)$$

with F as in (5.81); moreover, $F(\vartheta n, n, \vartheta) = \lim_{j \rightarrow (\vartheta n)^-} F(j, n, \vartheta)$. If $j < \vartheta n$, we have $s \neq +\infty$ and one obtains (5.85) using Lemma B.1 in Appendix B to evaluate $\int_{\mathbb{R}^d} dk \left(\frac{|k|^j}{\sqrt{1 + |k|^{2n}}} \right)^s$ ⁽¹⁹⁾. If $j = \vartheta n$ we have $s = +\infty$; $\frac{|\mathbf{k}|^{\vartheta n}}{\sqrt{1 + |\mathbf{k}|^{2n}}}$ is clearly continuous and vanishing at infinity, hence in \mathcal{L}^∞ , and $\left\| \frac{|\mathbf{k}|^{\vartheta n}}{\sqrt{1 + |\mathbf{k}|^{2n}}} \right\|_{\mathcal{L}^\infty} = \sup_{\eta \in [0, +\infty)} \frac{\eta^\vartheta}{\sqrt{1 + \eta^2}} = \sqrt{(1 - \vartheta)^{1 - \vartheta} \vartheta^\vartheta} = F(j, n, \vartheta)$ (the sup is attained at $\eta = \sqrt{\vartheta/(1 - \vartheta)}$). Finally, the statement $F(\vartheta n, n, \vartheta) = \lim_{j \rightarrow (\vartheta n)^-} F(j, n, \vartheta)$ is checked expressing $\ln F(j, n, \vartheta)$ via the Stirling formula $\ln \Gamma(z) = (z - 1/2) \ln z - z + (1/2) \ln(2\pi) + 1/(12z) + O(1/z^2)$ for $z \rightarrow +\infty$.

Step 3. The Sobolev inequality (5.12) holds, and its sharp constant $S(j, n, \vartheta)$ fulfills

$$S(j, n, \vartheta) \leq S_{++}(j, n, \vartheta) \quad (5.86)$$

with $S_{++}(j, n, \vartheta)$ as in Eq. (5.79). Let us keep in mind the definitions of r', s in Step 1, and write down the Hausdorff-Young inequality (2.19) with p, p' replaced by r', r ; this gives (expressing r, r' in terms of j, n, ϑ),

$$g \in \mathcal{L}^{r'} \quad \Rightarrow \quad \mathcal{F}^{-1}g \in \mathcal{L}^r, \quad \|\mathcal{F}^{-1}g\|_{\mathcal{L}^r} \leq \frac{E(j, n, \vartheta)}{\pi^{\vartheta n - j}} \|g\|_{\mathcal{L}^{r'}} \quad (5.87)$$

with E as in (5.80). Let us consider a function $f \in H^n$; writing $D^j f = \mathcal{F}^{-1}(|\mathbf{k}|^j \mathcal{F}f)$ and using (5.87) with $g = |\mathbf{k}|^j \mathcal{F}f$, we obtain

$$D^j f \in \mathcal{L}^r, \quad \|D^j f\|_{\mathcal{L}^r} \leq \frac{E(j, n, \vartheta)}{\pi^{\vartheta n - j}} \| |\mathbf{k}|^j \mathcal{F}f \|_{\mathcal{L}^{r'}} , \quad (5.88)$$

provided that $|\mathbf{k}|^j \mathcal{F}f \in \mathcal{L}^{r'}$. In order to check this last statement, we write

$$|\mathbf{k}|^j \mathcal{F}f = \frac{|\mathbf{k}|^j}{\sqrt{1 + |\mathbf{k}|^{2n}}} \sqrt{1 + |\mathbf{k}|^{2n}} \mathcal{F}f$$

¹⁹To be precise, use item (i) of the cited lemma with $a = j + d/s = \vartheta n$, $b = n$ and $u = s/2 = d/2(\vartheta n - j)$.

and recall the second relation (5.82) connecting r', s ; this yields that $|\mathbf{k}|^j \mathcal{F}f$ is actually in $\mathcal{L}^{r'}$, with

$$\| |\mathbf{k}|^j \mathcal{F}f \|_{\mathcal{L}^{r'}} \leq \left\| \frac{|\mathbf{k}|^j}{\sqrt{1 + |\mathbf{k}|^{2n}}} \right\|_{\mathcal{L}^s} \|\sqrt{1 + |\mathbf{k}|^{2n}} \mathcal{F}f\|_{\mathcal{L}^2} \quad (5.89)$$

$$= \pi^{\vartheta n/2 - j/2} F(j, n, \vartheta) \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}$$

(as for the last equality, recall (5.5) and (5.85)). Inserting (5.89) into (5.88), we conclude

$$D^j f \in \mathcal{L}^r, \quad \|D^j f\|_{\mathcal{L}^r} \leq S_{++}(j, n, \vartheta) \sqrt{\|f\|_{\mathcal{L}^2}^2 + \|D^n f\|_{\mathcal{L}^2}^2}, \quad (5.90)$$

with $S_{++}(j, n, \vartheta)$ as in (5.79). This proves the inequality (5.12) and the upper bound (5.86) on its sharp constant.

Step 4. In the subcase $\vartheta = j/n$, where $S(j, n, \vartheta) = \sqrt{(1 - \vartheta)^{1 - \vartheta} \vartheta^\vartheta}$ due to Corollary 5.2, one has $S_{++}(j, n, \vartheta) = S(j, n, \vartheta)$. This is readily checked using the definition (5.79) of $S_{++}(j, n, \vartheta)$.

Step 5. In the subcase $\vartheta = j/n + d/2n$, where $S(j, n, \vartheta)$ equals the right hand side of Eq. (5.57), one has $S_{++}(j, n, \vartheta) = S(j, n, \vartheta)$. This statement is checked using the definition (5.79) of $S_{++}(j, n, \vartheta)$, and recalling that $\Gamma(z)\Gamma(1 - z) = \pi/\sin(\pi z)$. \square

5.15 Corollary. For j, n, ϑ, r as in (5.78), consider the Gagliardo-Nirenberg inequality (5.11)

$$H^n \subset \mathcal{L}^{2, r, j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G \|f\|_{\mathcal{L}^2}^{1 - \vartheta} \|D^n f\|_{\mathcal{L}^2}^\vartheta \quad \text{for all } f \in H^n.$$

This holds and its sharp constant $G(j, n, \vartheta)$ has the upper bound

$$G(j, n, \vartheta) \leq G_{++}(j, n, \vartheta), \quad G_{++}(j, n, \vartheta) := \frac{S_{++}(j, n, \vartheta)}{\sqrt{(1 - \vartheta)^{1 - \vartheta} \vartheta^\vartheta}}, \quad (5.91)$$

with $S_{++}(j, n, \vartheta)$ as in Eq. (5.79).

The equality $G_{++}(j, n, \vartheta) = G(j, n, \vartheta)$ holds in the cases:

- (a) $j = \vartheta n$, where $G(j, n, \vartheta) = 1$ due to Proposition 5.1;
- (b) $\vartheta = j/n + d/2n$ (with $n \neq 0$), where $G(j, n, \vartheta) = \frac{1}{\sqrt{(1 - \vartheta)^{1 - \vartheta} \vartheta^\vartheta}} \times$ right hand side of Eq. (5.57).

Proof. For $n \neq 0$, everything follows from Propositions 5.14 and 3.8. Let us pass to the case $n = 0$, implying $j = 0$ due to (5.78); then $j = \vartheta n$ and (5.55) holds with $G(j, n, \vartheta) = 1$ due to Proposition 5.1. On the other hand, in this case $G_{++}(j, n, \vartheta) = 1$ due to the definitions (5.91) and (5.79). \square

Let us remark that, due to the first Γ function term in (5.81), the upper bound $S_{++}(j, n, \vartheta)$ of (5.79) diverges in the limit $\vartheta \rightarrow 1$; the same happens for the bound $G_{++}(j, n, \vartheta)$ in (5.91). On the other hand the special case $\vartheta = 1$, excluded from the present conditions (5.78), has been already studied in subsection 5.3.

5.6 Putting together the results of subsections 5.3, 5.5: upper bounds for the general \mathcal{L}^2 case

In the cited subsections we have proved that the Gagliardo-Nirenberg and Sobolev inequalities hold for j, n, ϑ as in (5.46) or for j, n, ϑ as in (5.78). The union of these two cases is just the “general \mathcal{L}^2 case” described by Eq. (5.9), i.e.,

$$0 \leq \vartheta \leq 1, \quad 0 \leq n, j < +\infty, \quad 0 \leq \vartheta n - j \leq \frac{d}{2}, \quad \vartheta \neq 1 \text{ if } n = j + \frac{d}{2};$$

$$\frac{1}{r} = \frac{1}{2} - \frac{\vartheta n - j}{d}.$$

Summing up, we have the following result.

5.16 Proposition. *The Gagliardo-Nirenberg and Sobolev inequalities hold in the general \mathcal{L}^2 case (5.9).*

Let us consider the sharp constants $G(j, n, \vartheta)$, $S(j, n, \vartheta)$. In the “almost general” case (5.46) we have for them the upper bounds $G_+(j, n, \vartheta)$, $S_+(j, n, \vartheta)$ of Proposition 5.9 and Corollary 5.10, that coincide with the sharp constants for $\vartheta = 1$ and diverge for ϑ approaching the limit value $j/n + d/2n$ (excluded from (5.46)). In the other “almost general” case (5.78), we have the upper bounds $G_{++}(j, n, \vartheta)$, $S_{++}(j, n, \vartheta)$ of Proposition 5.14 and Corollary 5.15, that coincide with the sharp constants for $\vartheta = j/n + d/2n$ and diverge for ϑ approaching the limit value 1 (excluded from (5.78)). Let us recall that the $+$ and $++$ bounds agree with the sharp constants even in the elementary case $j = \vartheta n$.

Due to the above features, one expects $G_+(j, n, \vartheta)$, $S_+(j, n, \vartheta)$ to be better (i.e., smaller) than $G_{++}(j, n, \vartheta)$, $S_{++}(j, n, \vartheta)$ for ϑ close to 1, and the contrary to happen for ϑ close to $j/n + d/2n$. This is confirmed by the numerical values reported in the next section for some sample choices of j, n, ϑ (in space dimension $d = 1, 2, 3$: see pages 48-49). Indeed, in these tests the $+$ bounds are better than the $++$ bounds only for ϑ very close to 1.

To conclude these considerations, let us recall that the $+$ and $++$ bounds agree with the sharp constants even in the elementary case $j = \vartheta n$ (with $n \neq 0$, if one considers the Sobolev inequality).

5.7 Lower bounds for the sharp constants in the general \mathcal{L}^2 case

Let us still refer to the general \mathcal{L}^2 framework of Eq. (5.9). There is an obvious strategy to obtain lower bounds on the sharp constants $G(j, n, \vartheta)$ and $S(j, n, \vartheta)$: one chooses any function $h \in H^n \setminus \{0\}$, hereafter referred to as a “trial function”, and notes that

$$G(j, n, \vartheta) \geq \frac{\|D^j h\|_{\mathcal{L}^r}}{\|h\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n h\|_{\mathcal{L}^2}^\vartheta} , \quad (5.92)$$

$$S(j, n, \vartheta) \geq \frac{\|D^j h\|_{\mathcal{L}^r}}{\sqrt{\|h\|_{\mathcal{L}^2}^2 + \|D^n h\|_{\mathcal{L}^2}^2}} ; \quad (5.93)$$

of course, in the choice of h one should try to make the right hand side of Eq. (5.92) or (5.93) as large as possible. Hereafter we present alternative lower bounds which cover the general case (5.9), and become very accurate for ϑ close to 1. These are obtained using as a trial function for the Gagliardo-Nirenberg inequality an approximant of the maximizer given by (5.43) for the case $\vartheta = 1$. From Corollary 5.6, we know that the maximizer in (5.43) is in \dot{H}^n , but it can fail to be in H^n ; the approximant introduced hereafter is a regularization, depending on a parameter $\varepsilon > 0$, that certainly belongs to H^n . Here is the statement implementing these ideas.

5.17 Proposition. *Let j, n, ϑ, r be as in Eq. (5.9). The sharp constant $G(j, n, \vartheta)$ in the Gagliardo-Nirenberg inequality (5.11) has the lower bound*

$$G(j, n, \vartheta) \geq G_-(j, n, \vartheta | \varepsilon) \quad \text{for all } \varepsilon \in (0, +\infty), \quad (5.94)$$

which obviously implies

$$G(j, n, \vartheta) \geq G_-(j, n, \vartheta) := \sup_{\varepsilon \in (0, +\infty)} G_-(j, n, \vartheta | \varepsilon) . \quad (5.95)$$

Here we have put

$$G_-(j, n, \vartheta | \varepsilon) := \frac{Y(j, n, \vartheta | \varepsilon)}{U(j, n | \varepsilon)^{1-\vartheta} V(j, n | \varepsilon)^\vartheta} , \quad (5.96)$$

where

$$U(j, n | \varepsilon) := \left(\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\xi \frac{\xi^{2n-2j+d-1} K_{n-j}^2(\xi)}{(\xi^2 + \varepsilon^2)^{2n-j}} \right)^{1/2} , \quad (5.97)$$

$$V(j, n | \varepsilon) := \left(\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\xi \frac{\xi^{4n-2j+d-1} K_{n-j}^2(\xi)}{(\xi^2 + \varepsilon^2)^{2n-j}} \right)^{1/2} \quad (5.98)$$

(K indicates, as usual, the Macdonald function); moreover

$$Y(j, n, \vartheta | \varepsilon) := \left(\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\rho \rho^{d-1} |M_{jn\varepsilon}(\rho)|^r \right)^{1/r}, \quad (5.99)$$

where $M_{jn\varepsilon} \in C([0, +\infty), \mathbb{R})$ is defined by

$$M_{jn\varepsilon}(\rho) := \int_0^{+\infty} d\xi \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} \frac{\xi^{n+d-1} K_{n-j}(\xi)}{(\xi^2 + \varepsilon^2)^{n-j/2}} \quad (5.100)$$

(intending $\left. \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} \right|_{\rho=0} := \lim_{s \rightarrow 0^+} \frac{J_{d/2-1}(s)}{s^{d/2-1}} = \frac{1}{2^{d/2-1}\Gamma(d/2)}$; note that Y depends on ϑ through r).

Proof. The idea is to apply the inequality (5.92) choosing for h the trial function

$$h_{jn\varepsilon} := \mathcal{F}^{-1} g_{jn\varepsilon}, \quad g_{jn\varepsilon} := \frac{|\mathbf{k}|^{n-j} K_{n-j}(|\mathbf{k}|)}{(|\mathbf{k}|^2 + \varepsilon^2)^{n-j/2}} \quad (\varepsilon > 0). \quad (5.101)$$

In the limit $\varepsilon \rightarrow 0$ this function becomes (up to a multiplicative constant) the maximizer (5.43) for the case $\vartheta = 1$ of the Gagliardo-Nirenberg inequality. In the sequel we fix any $\varepsilon > 0$ and analyze the features of the above function.

First of all, we note that $g_{jn\varepsilon}$ is in the space of continuous, rapidly decreasing functions on \mathbb{R}^d ; to check this, one should recall that $\lim_{\xi \rightarrow 0^+} \xi^\mu K_\mu(\xi)$ is finite for any $\mu > 0$, while $K_\mu(\xi) = O(e^{-\xi}/\sqrt{\xi})$ for $\xi \rightarrow +\infty$ [28]. It is clear that $g_{jn\varepsilon}, |\mathbf{k}|^n g_{jn\varepsilon} \in \mathcal{L}^2$, whence $h_{jn\varepsilon} \in H^n$; moreover, by the elementary rules for radial integrals (Eq. (B.3) of Appendix B),

$$\|h_{jn\varepsilon}\|_{\mathcal{L}^2} = \|g_{jn\varepsilon}\|_{\mathcal{L}^2} = U(j, n | \varepsilon) \quad \text{as in (5.97)}, \quad (5.102)$$

$$\|D^n h_{jn\varepsilon}\|_{\mathcal{L}^2} = \| |\mathbf{k}|^n g_{jn\varepsilon} \|_{\mathcal{L}^2} = V(j, n | \varepsilon) \quad \text{as in (5.98)}. \quad (5.103)$$

Let us pass to

$$D^j h_{jn\varepsilon} = \mathcal{F}^{-1} (|\mathbf{k}|^j g_{jn\varepsilon}) = \mathcal{F}^{-1} \frac{|\mathbf{k}|^n K_{n-j}(|\mathbf{k}|)}{(|\mathbf{k}|^2 + \varepsilon^2)^{n-j/2}}. \quad (5.104)$$

The general theory of radial Fourier transforms, see Appendix B ⁽²⁰⁾, gives

$$D^j h_{jn\varepsilon} = M_{jn\varepsilon}(|\mathbf{x}|) \quad M_{jn\varepsilon} \text{ as in (5.100)} \quad (5.105)$$

and also ensures that $M_{jn\varepsilon}$ is continuous on $[0, +\infty)$. The function $D^j h_{jn\varepsilon}$ is certainly in \mathcal{L}^r (this is established invoking the Gagliardo-Nirenberg inequality, or

²⁰In particular, Lemma B.2.

more directly via the Hausdorff-Young inequality: in fact $\mathcal{F}D^j h_{jn\varepsilon}$ is continuous and rapidly decreasing, hence in $\mathcal{L}^{r'}$ with $1/r + 1/r' = 1$). We have

$$\|h_{jn\varepsilon}\|_{\mathcal{L}^r} = Y(j, n | \varepsilon) \quad \text{as in (5.99)} . \quad (5.106)$$

In conclusion, the inequality (5.92) for the trial function $h_{jn\varepsilon}$ and Eqs. (5.96) (5.102) (5.103) (5.106) give

$$G(j, n, \vartheta) \geq \frac{\|D^j h_{jn\varepsilon}\|_{\mathcal{L}^r}}{\|h_{jn\varepsilon}\|_{\mathcal{L}^2}^{1-\vartheta} \|D^n h_{jn\varepsilon}\|_{\mathcal{L}^2}^{\vartheta}} = \frac{Y(j, n, \vartheta | \varepsilon)}{U(j, n | \varepsilon)^{1-\vartheta} V(j, n | \varepsilon)^{\vartheta}} = G_-(j, n, \vartheta | \varepsilon) ,$$

which is just the thesis (5.94). \square

5.18 Corollary. *Let j, n, ϑ, r be as in Eqs. (5.9), and $n \neq 0$. $S(j, n, \vartheta)$ has the lower bound*

$$S(j, n, \vartheta) \geq S_-(j, n, \vartheta | \varepsilon) \quad \text{for all } \varepsilon > 0, \quad (5.107)$$

which obviously implies

$$S(j, n, \vartheta) \geq S_-(j, n, \vartheta) := \sup_{\varepsilon \in (0, +\infty)} S_-(j, n, \vartheta | \varepsilon) ; \quad (5.108)$$

here we have put

$$S_-(j, n, \vartheta | \varepsilon) := \sqrt{(1 - \vartheta)^{1-\vartheta} \vartheta^{\vartheta}} G_-(j, n, \vartheta | \varepsilon), \quad G_- \text{ as in (5.96)}. \quad (5.109)$$

Proof. Use Propositions 5.17 and 3.8. \square

The previous Corollary excludes the trivial case $n = 0$ of Eq. (5.9), implying $j = 0$; let recall that $S(0, 0, \vartheta) = 1/\sqrt{2}$ due to Remark 5.3.

The forthcoming Proposition 5.19 (with its Corollary 5.20) presents an alternative lower bound for $S(j, n, \vartheta)$ (and its equivalent for $G(j, n, \vartheta)$), holding under certain conditions on ϑ ; in this case we use (5.93) choosing $h = f_{\lambda}$, where f is the maximizer of the special case $\vartheta = j/n + d/2n$ (see Eq. (5.58)) and the scaling parameter λ is determined so as to maximize the right hand side of (5.93). By construction, this lower bound is accurate when ϑ is close to the special value $j/n + d/2n$.

5.19 Proposition. *Let*

$$0 < n < +\infty, \quad 0 \leq j \leq n, \quad \frac{j}{n} \leq \vartheta \leq \frac{j}{n} + \frac{d}{2n} < 1; \quad (5.110)$$

define r as usual, via Eq. (5.9). $S(j, n, \vartheta)$ in Eq. (5.12) has the lower bound

$$S(j, n, \vartheta) \geq S_{--}(j, n, \vartheta) , \quad (5.111)$$

$$S_{--}(j, n, \vartheta) := \frac{I(j, n, \vartheta)}{\pi^{d/4+1/2}} \sqrt{\frac{\Gamma(d/2)n \sin(\pi(j/n + d/2n))(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta}{(1-j/n-d/2n)^{1-\vartheta}(j/n+d/2n)^\vartheta}} ;$$

here

$$I(j, n, \vartheta) := \begin{cases} \frac{\pi}{2^{d/2}\Gamma(d/2)n \sin(\pi\vartheta)} & \text{if } \vartheta = j/n + d/2n \\ \left(\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\rho \rho^{d-1} |L_{jn}(\rho)|^r \right)^{1/r} & \text{if } \vartheta \neq j/n + d/2n \end{cases} \quad (5.112)$$

and $L_{jn} \in C([0, +\infty), \mathbb{R})$ is defined in terms of the Bessel function $J_{d/2-1}$ by

$$L_{jn}(\rho) := \int_0^{+\infty} d\xi \frac{J_{d/2-1}(\rho\xi) \xi^{d+2j-1}}{(\rho\xi)^{d/2-1} \frac{1}{1+\xi^{2n}}} \quad \text{for } \rho > 0, \quad (5.113)$$

$$L_{jn}(0) = \frac{\pi}{2^{d/2}\Gamma(d/2)n \sin(\pi \frac{2j+d}{2n})} .$$

L_{jn} can be expressed in terms of the Fox H -function or, if n is rational, in terms of the Meijer G -function; the expressions for L_{jn} via H or G are equal to the ones given for the function F_{jn} in Eqs. (5.60)-(5.63), with j replaced systematically by $2j$. The equality $S(j, n, \vartheta) = S_{--}(j, n, \vartheta)$ holds in the following cases:

- (a) $\vartheta = 0$, implying $j = 0$, $r = 2$ (due to (5.110)) and $S(j, n, \vartheta) = 1$ (due to Corollary 5.2);
- (b) $\vartheta = j/n + d/2n$, where $S(j, n, \vartheta) =$ right hand side of Eq. (5.57).

Proof. Step 1. One has

$$S(j, n, \vartheta) \geq S_{--}(j, n, \vartheta) \quad (5.114)$$

with $S_{--}(j, n, \vartheta)$ as in Eq. (5.111); moreover the function L_{jn} of Eq. (5.113), related to the definition of $S_{--}(j, n, \vartheta)$, has all the features claimed in the text of the proposition. As anticipated, this result is obtained using the maximizer of the special case $\vartheta = j/n + d/2n$, i.e., the H^n function

$$f := \mathcal{F}^{-1} \left(\frac{|\mathbf{k}|^j}{1 + |\mathbf{k}|^{2n}} \right)$$

of Eqs. (5.58) (5.59). For any ϑ as in (5.110), we apply (5.93) with the trial function $h := f_\lambda$, with $\lambda > 0$ unspecified for the moment. This gives

$$S(j, n, \vartheta) \geq \frac{\|D^j f_\lambda\|_{\mathcal{L}^r}}{\sqrt{\|f_\lambda\|_{\mathcal{L}^2}^2 + \|D^n f_\lambda\|_{\mathcal{L}^2}^2}} \quad (5.115)$$

$$= \frac{\|D^j f\|_{\mathcal{L}^r}}{\sqrt{\lambda^{-2\vartheta n} \|f\|_{\mathcal{L}^2}^2 + \lambda^{2(1-\vartheta)n} \|D^n f\|_{\mathcal{L}^2}^2}} \quad \text{for all } \lambda > 0 ,$$

the last equality following from (3.13). To go on, we note that

$$\|f\|_{\mathcal{L}^2} = \left\| \frac{|\mathbf{k}|^j}{1 + |\mathbf{k}|^{2n}} \right\|_{\mathcal{L}^2} = \sqrt{A(j, n)}, \quad (5.116)$$

$$A(j, n) := \frac{\pi^{d/2+1} (1 - j/n - d/2n)}{\Gamma(d/2) n \sin(\pi(j/n + d/2n))};$$

$$\|D^n f\|_{\mathcal{L}^2} = \left\| \frac{|\mathbf{k}|^{j+n}}{1 + |\mathbf{k}|^{2n}} \right\|_{\mathcal{L}^2} = \sqrt{B(j, n)}, \quad (5.117)$$

$$B(j, n) := \frac{\pi^{d/2+1} (j/n + d/2n)}{\Gamma(d/2) n \sin(\pi(j/n + d/2n))}.$$

For the expression of the above integrals in terms of $A(j, n)$ and $B(j, n)$, see Lemma B.1 in Appendix B ⁽²¹⁾. Moreover $D^j f = \mathcal{F}^{-1} \left(\frac{|\mathbf{k}|^{2j}}{1 + |\mathbf{k}|^{2n}} \right)$ is in \mathcal{L}^∞ and admits the representation

$$D^j f(x) = L_{jn}(|x|) \quad \text{for } x \in \mathbb{R}^d, \quad L_{jn} \in C([0, +\infty), \mathbb{R}) \quad \text{as in (5.113)}; \quad (5.118)$$

for this statement we refer to Lemma B.3 which also justifies the expression of L_{jn} as in Eqs. (5.60)-(5.63), with j replaced by $2j$ ⁽²²⁾. If $\vartheta \neq j/n + d/2n$ we have $r \neq \infty$, so

$$\|D^j f\|_{\mathcal{L}^r} = \left(\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\rho \rho^{d-1} |L_{jn}(\rho)|^r \right)^{1/r}; \quad (5.119)$$

if $\vartheta = j/n + d/2n$ we have $r = \infty$ and, due to Lemma B.3, Eq. (B.29),

$$\|D^j f\|_{\mathcal{L}^\infty} = (D^j f)(0) = \frac{\pi}{2^{d/2} \Gamma(d/2) n \sin(\pi\vartheta)}. \quad (5.120)$$

In both cases, we can write

$$\|D^j f\|_{\mathcal{L}^r} = I(j, n, \vartheta) \quad \text{as in Eq. (5.112)}. \quad (5.121)$$

Inserting Eqs. (5.116) (5.117) (5.121) into (5.115) we obtain

$$S(j, n, \vartheta) \geq \frac{I(j, n, \vartheta)}{\sqrt{\lambda^{-2\vartheta n} A(j, n) + \lambda^{2(1-\vartheta)n} B(j, n)}} \quad \text{for all } \lambda > 0, \quad (5.122)$$

which implies

$$S(j, n, \vartheta) \geq \frac{I(j, n, \vartheta)}{\sqrt{\inf_{\lambda \in (0, +\infty)} (\lambda^{-2\vartheta n} A(j, n) + \lambda^{2(1-\vartheta)n} B(j, n))}}. \quad (5.123)$$

²¹Use this lemma with the following choices: $a = d/4 + j/2$, $b = n$, $u = 2$ to get Eq. (5.116), $a = d/4 + j/2 + n/2$, $b = n$, $u = 2$ to get Eq. (5.117).

²²Use this lemma with $a = j + d/2$ and $b = n$.

The above inf equals $\frac{A(j,n)^{1-\vartheta} B(j,n)^\vartheta}{(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta}$ and is attained at $\lambda = \left(\frac{\vartheta A(j,n)}{(1-\vartheta) B(j,n)} \right)^{1/2n}$ (for $\vartheta = 0$, the last statement means that the inf is the $\lambda \rightarrow 0^+$ limit of the function). In conclusion,

$$S(j, n, \vartheta) \geq I(j, n, \vartheta) \sqrt{\frac{(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta}{A(j, n)^{1-\vartheta} B(j, n)^\vartheta}} = S_{--}(j, n, \vartheta) , \quad (5.124)$$

where the last equality follows expressing $A(j, n), B(j, n)$ via Eqs. (5.116) (5.117) and comparing with the definition (5.111) of $S_{--}(j, n, \vartheta)$.

Step 2. If $\vartheta = 0$, Eq. (5.9) implies $j = 0, r = 2$ due to (5.9) and one has $S(j, n, \vartheta) = 1$ due to Corollary 5.2; moreover $S_{--}(j, n, \vartheta) = 1$. Let $\vartheta = 0$. It is obvious that (5.9) implies $j = 0, r = 2$ and that the cited corollary gives $S(0, n, 0) = 1$. To go on, we note that Eq. (5.124) gives $S_{--}(0, n, 0) := I(0, n, 0)/\sqrt{A(0, n)}$; on the other hand, Eqs. (5.116) (5.121) imply $\sqrt{A(0, n)} = \|f\|_{\mathcal{L}^2} = I(0, n, 0)$, so $S_{--}(0, n, 0) = 1$.

Step 3. If $\vartheta = j/n + d/2n$, one has $S(j, n, \vartheta) = S_{--}(j, n, \vartheta) =$ the right hand side of Eq. (5.57). This statement is checked by direct comparison between the definitions (5.111) of $S_{--}(j, n, \vartheta)$ and Eq. (5.57). An alternative proof is as follows: the derivation of the bound $S(j, n, \vartheta) \geq S_{--}(j, n, \vartheta)$ given in Step 1 is reduced, in the special case $\vartheta = j/n + d/2n$, to the derivation of the bound $S(j, n, j/n + d/2n) \geq$ right hand side of (5.57), given in Step 2 within the proof of Proposition 5.11. (Note that if f is the function defined by Eq. (5.58) the fact that f is a maximizer for $\vartheta = j/n + d/2n$ automatically ensures that in this subcase the ratio (5.115) attains its maximum at $\lambda = 1$.) Thus $S_{--}(j, n, j/n + d/2n) =$ right hand side of (5.57); on the other hand, from Proposition 5.11 we already know that $S(j, n, j/n + d/2n) =$ right hand side of (5.57), whence the thesis. \square

5.20 Corollary. For j, n, ϑ, r as in (5.110), $G(j, n, \vartheta)$ fulfills the bounds

$$G(j, n, \vartheta) \geq G_{--}(j, n, \vartheta) , \quad G_{--}(j, n, \vartheta) := \frac{S_{--}(j, n, \vartheta)}{\sqrt{(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta}} , \quad (5.125)$$

with $S_{--}(j, n, \vartheta)$ as in Eq. (5.111).

The equality $G(j, n, \vartheta) = G_{--}(j, n, \vartheta)$ holds in the cases:

- (a) $\vartheta = 0$, where $j = 0, r = 2$ (due to (5.110)) and $G(j, n, \vartheta) = 1$ (due to Proposition 5.1);
- (b) $\vartheta = j/n + d/2n$, where $G(j, n, \vartheta) = \frac{1}{\sqrt{(1-\vartheta)^{1-\vartheta} \vartheta^\vartheta}} \times$ right hand side of Eq. (5.57).

Proof. Use Propositions 5.19 and 3.8. \square

Comparison between the previous lower bounds. Let us repeat that the lower bounds $G_{--}(j, n, \vartheta), S_{--}(j, n, \vartheta)$ are defined under the condition (5.110) (containing

the essential limitation $j/n + d/2n < 1$), while $G_-(j, n, \vartheta)$, $S_-(j, n, \vartheta)$ are defined in the general \mathcal{L}^2 case (5.9). When both lower bounds $-$ and $--$ are defined, one naturally wonders which one is better (i.e., larger). This depends on the choice of j, n, ϑ : see, for example, the cases considered in Table III, page 49 (and the introduction to this table on page 48).

6 Some examples

Sharp constants and maximizers for the Sobolev and Gagliardo-Nirenberg inequalities, in the \mathcal{L}^∞ subcase (5.53) - (5.56). The forthcoming Table I reports the values of $\vartheta(j, n)$ and of the sharp constants $S(j, n)$, $G(j, n)$ obtained from Eqs. (5.53) (5.57) (5.77), for $d = 1, 2, 3$ and some choices of j, n . In this table, many lines consider pairs (j, n) and (j', n) such that $\vartheta(j, n) + \vartheta(j', n) = 1$, so that $S(j, n) = S(j', n)$ and $G(j, n) = G(j', n)$ (see Remark 5.13). For example, the second line refers to the pairs $(0, 3/2)$ and $(1/2, 3/2)$, such that $\vartheta(0, 3/2) = 1/3$ and $\vartheta(1/2, 3/2) = 2/3$.

We know that, for j, n as in (5.53), the inequality (5.55) or (5.56) admits a maximizer $f = F_{jn}(|\mathbf{x}|)$, where $F_{jn} : [0, +\infty) \rightarrow \mathbb{R}$, $\rho \mapsto F_{jn}(\rho)$ is defined by Eq. (5.59) in terms of an integral involving a Bessel function. In a few cases the integral is elementary, otherwise F_{jn} can be expressed in terms of a Fox H -function or Meijer G -function as in Eqs. (5.60)-(5.63). Table II reports, as examples, the functions F_{jn} for $d = 1, 2, 3$ and some choices of (j, n) . The parameters of the G -function for the cases appearing in Table II have been determined using Eqs. (5.62) (5.63) and also taking into account the symmetry properties mentioned in Appendix B; these allow to rearrange the sequences of parameters (a_ℓ) , (b_ℓ) , (b_ℓ^*) of Eq. (5.63) in increasing order and to eliminate from the parameters any pair $(a_{\ell_1}, b_{\ell_2}^*)$ with $a_{\ell_1} = b_{\ell_2}^*$.

Table I. Values of $\vartheta(j, n)$ and sharp constants of the inequalities (5.55)(5.56), given by (5.53)(5.57)(5.77): some examples.

d	(j, n)	$\vartheta(j, n)$	$S(j, n)$	$G(j, n)$
1	$(0, 1)$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = 0.707...$	1
	$(0, \frac{3}{2})$ or $(\frac{1}{2}, \frac{3}{2})$	$\frac{1}{3}$ or $\frac{2}{3}$	$\frac{\sqrt{2}}{3^{3/4}} = 0.620...$	$\frac{2^{1/6}}{3^{1/4}} = 0.852...$
	$(0, 2)$ or $(1, 2)$	$\frac{1}{4}$ or $\frac{3}{4}$	$\frac{1}{2^{3/4}} = 0.594...$	$\frac{2^{1/4}}{3^{3/8}} = 0.787...$
	$(4, 10)$ or $(5, 10)$	$\frac{9}{20}$ or $\frac{11}{20}$	$\frac{1}{2\sqrt{5}\sin(9\pi/20)} = 0.224...$	$\frac{1}{3^{9/20}11^{11/40}\sqrt{\sin(9\pi/20)}} = 0.317...$
2	$(0, \frac{3}{2})$	$\frac{2}{3}$	$\frac{1}{3^{3/4}} = 0.438...$	$\frac{1}{2^{1/3}3^{1/4}} = 0.603...$
	$(0, 2)$	$\frac{1}{2}$	$\frac{1}{2^{3/2}} = 0.353...$	$\frac{1}{2}$
	$(0, \frac{5}{2})$ or $(\frac{1}{2}, \frac{5}{2})$	$\frac{2}{5}$ or $\frac{3}{5}$	$\frac{1}{\sqrt{5}} \left(\frac{2}{5 + \sqrt{5}} \right)^{1/4} = 0.324...$	$\frac{2^{1/20}}{3^{3/10}(5 + \sqrt{5})^{1/4}} = 0.453...$
	$(0, 3)$ or $(1, 3)$	$\frac{1}{3}$ or $\frac{2}{3}$	$\frac{1}{\sqrt{2}3^{3/4}} = 0.310...$	$\frac{1}{2^{5/6}3^{1/4}} = 0.426...$
3	$(0, 2)$	$\frac{3}{4}$	$\frac{1}{2^{5/4}\sqrt{\pi}} = 0.237...$	$\frac{1}{2^{1/4}3^{3/8}\sqrt{\pi}} = 0.314...$
	$(0, \frac{5}{2})$	$\frac{3}{5}$	$\frac{1}{\sqrt{5}\pi} \left(\frac{2}{5 + \sqrt{5}} \right)^{1/4} = 0.182...$	$\frac{2^{1/20}}{3^{3/10}(5 + \sqrt{5})^{1/4}\sqrt{\pi}} = 0.256...$
	$(0, 3)$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}\pi} = 0.162...$	$\frac{1}{\sqrt{6}\pi} = 0.230...$
	$(1, 3)$	$\frac{5}{6}$	$\frac{1}{\sqrt{6}\pi} = 0.230...$	$\frac{1}{5^{5/12}\sqrt{\pi}} = 0.288...$

Table II. Maximizers for the inequalities (5.55) - (5.56), computed via (5.59) - (5.63): some examples.

d	(j, n)	$F_{jn}(\rho)$
1	$(0, 1)$	$\sqrt{\frac{\pi}{2}} e^{-\rho}$
	$(0, 2)$	$\frac{\sqrt{\pi}}{2} \left(\cos \frac{\rho}{\sqrt{2}} - \sin \frac{\rho}{\sqrt{2}} \right) \left(\cosh \frac{\rho}{\sqrt{2}} - \sinh \frac{\rho}{\sqrt{2}} \right)$
	$(1, 2)$	$\frac{1}{2} G \left(\begin{matrix} 1/2; \\ 0, 1/2, 1/2; 1/4, 3/4 \end{matrix} \middle \left(\frac{\rho}{4} \right)^4 \right)$
2	$(0, \frac{3}{2})$	$\frac{1}{6\pi} G \left(\begin{matrix} 1/6; \\ 0, 1/6, 1/3, 2/3, 2/3; 0, 1/3 \end{matrix} \middle \left(\frac{\rho}{6} \right)^6 \right)$
	$(0, 2)$	$\frac{1}{4} G \left(\begin{matrix} ; \\ 0, 1/2, 1/2; 0 \end{matrix} \middle \left(\frac{\rho}{4} \right)^4 \right)$
	$(1, 3)$	$\frac{1}{6} G \left(\begin{matrix} 1/2; \\ 0, 1/3, 1/2, 2/3; 0, 1/3, 2/3 \end{matrix} \middle \left(\frac{\rho}{6} \right)^6 \right)$
3	$(0, 2)$	$\sqrt{\frac{\pi}{2}} \frac{e^{-\rho/\sqrt{2}}}{\rho} \sin \frac{\rho}{\sqrt{2}}$
	$(0, 3)$	$\frac{1}{6\sqrt{6}} G \left(\begin{matrix} ; \\ 0, 1/3, 1/2, 2/3; -1/6, 1/6 \end{matrix} \middle \left(\frac{\rho}{6} \right)^6 \right)$
	$(1, 3)$	$\frac{1}{6\sqrt{6}} G \left(\begin{matrix} 1/3; \\ 0, 1/3, 1/3, 2/3; -1/6, 1/6, 1/2 \end{matrix} \middle \left(\frac{\rho}{6} \right)^6 \right)$

Upper and lower bounds for the sharp constant in the Gagliardo-Nirenberg inequality (5.11): numerical values in some examples. In the forthcoming Table III we present, for $d = 1, 2, 3$:

- (i) some choices of (j, n) ;
- (ii) the interval $\Theta_{jn} \equiv \Theta$ of the values of ϑ fulfilling the conditions (5.9) for the chosen pair (j, n) ;
- (iii) one or two sample choices for ϑ in Θ ;
- (iv) the value of $r(j, n, \vartheta) \equiv r$, see Eq. (5.9);
- (v) the numerical values of the lower bounds G_- , G_{--} (Eqs. (5.95) (5.125)) and of the upper bounds G_+ , G_{++} (Eqs. (5.47) (5.91)) for the sharp constants ($G_- \equiv G_-(j, n, \vartheta)$, \dots , $G_{++} \equiv G_{++}(j, n, \vartheta)$). To be more precise, the quantities indicated in Table III as G_- , \dots , G_{++} are lower or upper approximants obtained numerically for the theoretical bounds defined by (5.95) (5.125) (5.47) (5.91).

In the column about $G_{--}(j, n, \vartheta)$, blank boxes refer to cases where this bound is undefined because $j/n + d/2n \geq 1$. For each choice of (j, n, ϑ) , boldface is used to indicate the best available lower bound (i.e., the maximum between G_- and G_{--} , when both of them are defined) and the best available upper bound (i.e., the minimum between G_+ and G_{++}).

Let us add some information about the computation of the upper and lower bounds. Concerning G_+ and G_{++} , which have explicit expressions in terms of Gamma and elementary functions, we have calculated their numerical values, rounding up the results to the number of digits reported in the table.

To compute the lower bound G_- , which is defined maximizing with respect to a parameter $\varepsilon > 0$, we have operated in this way: firstly we have computed numerically all integrals in Eqs. (5.96)-(5.100) for a grid of sample values of ε (with spacing $1/100$), and then we have taken the maximum over the grid (rounding down to the digits reported in the table). This is, in any case, a lower bound for the sharp constant (in fact smaller, but not too much smaller than the sup in the definition (5.95) of G_-).

The lower bound G_{--} depends on the integral of a certain function $L_{jn} \equiv L$, see Eq. (5.113). For the choices of j, n under consideration, this function has been expressed as a Meijer G-function following the indications after Eq. (5.113); subsequently, its integral has been computed numerically (and the final value for G_{--} obtained in this way has been rounded down to the digits reported in the table).

All the numerical calculations mentioned before have been performed using **Mathematica**. For the choices of j, n, ϑ in the table, we have the following indications:

- (a) The upper bound G_{++} is generally better than G_+ ; G_+ is better only for ϑ very close to 1. This is in agreement with the anticipations given on page 4.
- (b) The best lower and upper bounds are generally close (or even very close), thus confining the sharp constant to a narrow interval. Less satisfactory results are obtained when j, n are large and close (see, e.g., the case $d = 1, j = 9, n = 10, \vartheta = 37/40$, in which the ratio lower bound/upper bound is, approximately, 0.53).

To conclude, we point out that it would be easy to produce an analogue of Table III for the Sobolev inequality (5.12), reporting the lower or upper bounds $S_\sigma(j, n, \vartheta) \equiv S_\sigma$ of the previous section for the sharp constants ($\sigma = -, --, +, ++$); let us recall that $S_\sigma(j, n, \vartheta) = \sqrt{(1-\vartheta)^{1-\vartheta}\vartheta^\vartheta} G_\sigma(j, n, \vartheta)$ for all σ . The previous comments about G_σ (in particular, statements (a) (b)) could be repeated for the bounds S_σ .

Table III. On the Gagliardo-Nirenberg inequality (5.11), in some examples with $d = 1, 2, 3$. G_-, G_{--} are the lower bounds (5.95) (5.125) on the sharp constants, G_+, G_{++} are the upper bounds (5.47) (5.91) (all of them depending on j, n, ϑ). Boldface is used to indicate the best lower and upper bounds; see page 48 for more indications.

d	(j, n)	Θ	ϑ	r	G_-	G_{--}	G_+	G_{++}
1	(0, 1)	$\{0 \leq \vartheta \leq 1/2\}$	1/3	6	0.849	0.832	1.204	0.873
	(3/4, 1)	$\{3/4 \leq \vartheta \leq 1\}$	9/10	20/7	0.867		1.030	0.944
	(3/4, 1)	$\{3/4 \leq \vartheta \leq 1\}$	99/100	50/13	0.950		1.078	1.564
	(1, 2)	$\{1/2 \leq \vartheta \leq 3/4\}$	5/8	4	0.608	0.633	1.087	0.711
	(5, 10)	$\{1/2 \leq \vartheta \leq 11/20\}$	21/40	4	0.080	0.421	1.087	0.471
	(9, 10)	$\{9/10 \leq \vartheta \leq 19/20\}$	37/40	4	0.317	0.00894	1.087	0.592
2	(0, 2)	$\{0 \leq \vartheta \leq 1/2\}$	1/3	6	0.504	0.498	0.741	0.511
	(0, 3)	$\{0 \leq \vartheta \leq 1/3\}$	1/6	4	0.533	0.547	0.752	0.554
	(1/2, 1)	$\{1/2 \leq \vartheta \leq 1\}$	3/4	8/3	0.766		0.848	0.782
	(1/2, 1)	$\{1/2 \leq \vartheta \leq 1\}$	9/10	10/3	0.714		0.781	0.795
	(1, 3)	$\{1/3 \leq \vartheta \leq 2/3\}$	5/9	6	0.387	0.414	0.741	0.436
	(9, 10)	$\{9/10 \leq \vartheta < 1\}$	19/20	4	0.359		0.752	0.504
3	(0, 2)	$\{0 \leq \vartheta \leq 3/4\}$	3/8	4	0.389	0.359	0.494	0.394
	(0, 3)	$\{0 \leq \vartheta \leq 1/2\}$	1/3	6	0.273	0.278	0.428	0.284
	(1, 3)	$\{1/3 \leq \vartheta \leq 5/6\}$	2/3	6	0.264	0.250	0.428	0.284
	(2, 3)	$\{2/3 \leq \vartheta \leq 1\}$	95/100	60/13	0.385		0.461	0.453
	(2, 3)	$\{2/3 \leq \vartheta \leq 1\}$	99/100	300/53	0.396		0.433	0.677
	(9, 10)	$\{9/10 \leq \vartheta \leq 1\}$	19/20	3	0.321		0.609	0.469

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A Appendix. Proof of Proposition 3.4

Let $p, q \in [1, +\infty]$, $j, n \in [0, +\infty)$, and assume Eq. (3.7)

$$0 \leq \frac{1}{r} = \frac{1}{q} - \frac{n-j}{d} \leq 1 .$$

The Gagliardo-Nirenberg inequality of order $(p, q; j, n, \vartheta = 1)$ is statement (3.8), reported hereafter:

$$\mathcal{L}^{p,q,n} \subset \mathcal{L}^{p,r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G \|D^n f\|_{\mathcal{L}^q} \quad \text{for some } G \in [0, +\infty) \text{ and all } f \in \mathcal{L}^{p,q,n}.$$

The corresponding extended Gagliardo-Nirenberg inequality is statement (3.9), that we also report:

$$\dot{\mathcal{L}}^{q,n} \subset \dot{\mathcal{L}}^{r,j}, \quad \|D^j f\|_{\mathcal{L}^r} \leq G \|D^n f\|_{\mathcal{L}^q} \quad \text{for some } G \in [0, +\infty) \text{ and all } f \in \dot{\mathcal{L}}^{q,n}.$$

Let us recall the notations $G(p, q; j, n)$ and $G(q; j, n)$ for the corresponding sharp constants. Proposition 3.4 states the equivalence of these inequalities and the equality of their sharp constants for $q \neq 1, +\infty$.

Proof of Proposition 3.4. *Step 1.* The extended inequality (3.9) implies the inequality (3.8), with $G(p, q; j, n) \leq G(q; j, n)$. This follows immediately by comparing the two statements under analysis and taking into account that $\mathcal{L}^{p,q,n} = \mathcal{L}^p \cap \dot{\mathcal{L}}^{q,n}$, $\mathcal{L}^{p,r,j} = \mathcal{L}^p \cap \dot{\mathcal{L}}^{r,j}$.

Step 2. The inequality (3.8) implies the extended inequality (3.9), with $G(q; j, n) \leq G(p, q; j, n)$. Let us assume the inequality (3.8), that we write hereafter using its sharp constant:

$$\mathcal{L}^{p,q,n} \subset \mathcal{L}^{p,r,j}, \quad \|D^j f'\|_{\mathcal{L}^r} \leq G(p, q; j, n) \|D^n f'\|_{\mathcal{L}^q} \quad \text{for all } f' \in \mathcal{L}^{p,q,n}. \quad (\text{A.1})$$

Let us consider a function

$$f \in \dot{\mathcal{L}}^{q,n}, \quad (\text{A.2})$$

which is fixed in the sequel. By the density of $\mathcal{L}^{p,q,n}$ in $\dot{\mathcal{L}}^{q,n}$ (see the final lines of Section 2), there is a sequence

$$f_\ell \in \mathcal{L}^{p,q,n} \quad (\ell \in \mathbb{N}) \quad (\text{A.3})$$

such that, in the limit $\ell \rightarrow \infty$,

$$D^n f_\ell \rightarrow D^n f \text{ in } \mathcal{L}^q . \quad (\text{A.4})$$

Since $\mathcal{L}^q \hookrightarrow \Phi'$ we also have $D^n f_\ell \rightarrow D^n f$ in Φ' ; applying to this relation the continuous map $D^{-n} : \Phi' \rightarrow \Phi'$ we obtain

$$f_\ell \rightarrow f \text{ in } \Phi' . \quad (\text{A.5})$$

To go on, let us write the inequality in (A.1) with $f' = f_\ell$ or $f' = f_\ell - f_{\ell'}$, for $\ell, \ell' \in \mathbb{N}$; this gives the following:

$$\|D^j f_\ell\|_{\mathcal{L}^r} \leq G(p, q; j, n) \|D^n f_\ell\|_{\mathcal{L}^q} \quad (\text{A.6})$$

$$\|D^j f_\ell - D^j f_{\ell'}\|_{\mathcal{L}^r} \leq G(p, q; j, n) \|D^n f_\ell - D^n f_{\ell'}\|_{\mathcal{L}^q} . \quad (\text{A.7})$$

Eqs. (A.4) and (A.7) imply

$$\|D^j f_\ell - D^j f_{\ell'}\|_{\mathcal{L}^r} \rightarrow 0 \quad \text{for } \ell, \ell' \rightarrow +\infty , \quad (\text{A.8})$$

yielding the existence of

$$f^j := \lim_{\ell \rightarrow +\infty} D^j f_\ell \text{ in } \mathcal{L}^r . \quad (\text{A.9})$$

On the other hand, Eq. (A.5) and the continuity of $D^j : \Phi' \rightarrow \Phi'$ give

$$D^j f_\ell \rightarrow D^j f \text{ in } \Phi' ; \quad (\text{A.10})$$

noting that (A.9) implies $D^j f_\ell \rightarrow f^j$ in Φ' , by comparison with (A.10) we obtain $f^j = D^j f$ and, returning to (A.9),

$$D^j f = \lim_{\ell \rightarrow +\infty} D^j f_\ell \text{ in } \mathcal{L}^r . \quad (\text{A.11})$$

Now, sending ℓ to $+\infty$ in Eq. (A.6) and recalling Eqs. (A.4) (A.11), we obtain

$$\|D^j f\|_{\mathcal{L}^r} \leq G(p, q; j, n) \|D^n f\|_{\mathcal{L}^q} . \quad (\text{A.12})$$

These conclusions hold for an arbitrary $f \in \dot{\mathcal{L}}^{q,n}$; thus, the extended inequality (3.9) is true and its sharp constant is such that $G(q; j, n) \leq G(p, q; j, n)$.

Step 3. Conclusion of the proof. Steps 1 and 2 clearly give the thesis. \square

B Appendix. On certain integrals and inverse Fourier transforms. The H - and G -functions

Radial integrals. Let us consider a measurable function g on \mathbb{R}^d of the form

$$g(k) = G(|k|) \quad \text{for } k \in \mathbb{R}^d \setminus \{0\} , \quad (\text{B.1})$$

where

$$G : (0, +\infty) \rightarrow \mathbb{C} \text{ measurable and } \int_0^{+\infty} d\xi \xi^{d-1} |G(\xi)| < +\infty . \quad (\text{B.2})$$

Then, as well known, $g \in \mathcal{L}^1$ and

$$\int_{\mathbb{R}^d} dk g(k) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\xi \xi^{d-1} G(\xi) . \quad (\text{B.3})$$

(More generally, if g has the form (B.1) with a measurable $G : (0, +\infty) \rightarrow [0, +\infty)$, Eq. (B.3) holds with both sides possibly equal to $+\infty$).

From the above statement one infers the following Lemma, used in the main text; this refers to the Pochhammer symbol $(z)_\ell$ (see, e.g., [28]), which is defined as follows for $z \in \mathbb{C}$ and $\ell \in \{0, 1, 2, \dots\}$:

$$(z)_0 := 1 , \quad (z)_\ell := \prod_{i=0}^{\ell-1} (z + i) \text{ for } \ell = 1, 2, \dots . \quad (\text{B.4})$$

B.1 Lemma. (i) Let $a, b, u \in \mathbb{R}$, $b > a > 0$ and $u > 0$; one has

$$\int_{\mathbb{R}^d} dk \frac{|k|^{2au-d}}{(1 + |k|^{2b})^u} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{+\infty} d\xi \frac{\xi^{2au-1}}{(1 + \xi^{2b})^u} \quad (\text{B.5})$$

and

$$\int_0^{+\infty} d\xi \frac{\xi^{2au-1}}{(1 + \xi^{2b})^u} = \frac{\Gamma(au/b) \Gamma(u - au/b)}{2b \Gamma(u)} . \quad (\text{B.6})$$

(ii) In particular, let $a, b \in \mathbb{R}$ with $b > a > 0$ and $u \in \{1, 2, 3, \dots\}$; then

$$\int_0^{+\infty} d\xi \frac{\xi^{2au-1}}{(1 + \xi^{2b})^u} = \frac{\pi(1 - au/b)_{u-1}}{2b(u-1)! \sin(\pi au/b)} . \quad (\text{B.7})$$

In the special case $au/b = m \in \{1, 2, \dots, u-1\}$, the last formula should be applied understanding

$$\frac{\pi(1-m)_{u-1}}{\sin(\pi m)} := \lim_{\varepsilon \rightarrow 0} \frac{\pi(1-m-\varepsilon)_{u-1}}{\sin(\pi(m+\varepsilon))} = (-1)^{m+1} \prod_{i=0, i \neq m-1}^{u-2} (1-m+i) . \quad (\text{B.8})$$

Proof. (i) Eq. (B.5) follows from Eq. (B.3). Moreover, with a change of variable $\xi = t^{1/2b}$ we get

$$\int_0^{+\infty} d\xi \frac{\xi^{2au-1}}{(1+\xi^{2b})^u} = \frac{1}{2b} \int_0^{+\infty} dt \frac{t^{au/b-1}}{(1+t)^u} = \frac{\Gamma(au/b)\Gamma(u-au/b)}{2b\Gamma(u)}$$

(for the last statement see, e.g., [28], Eq. (5.12.3)); this proves Eq. (B.6).

(ii) Let us apply Eq. (B.6) with $u \in \{1, 2, 3, \dots\}$, assuming provisionally that au/b is noninteger. The known relations $\Gamma(z+\ell) = (z)_\ell \Gamma(z)$ (for $\ell \in \mathbb{N}$) and $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ (for z noninteger) give

$$\begin{aligned} \Gamma(au/b)\Gamma(u-au/b) &= \Gamma(au/b)\Gamma(1-au/b+(u-1)) \\ &= (1-au/b)_{u-1}\Gamma(au/b)\Gamma(1-au/b) = \frac{\pi(1-au/b)_{u-1}}{\sin(\pi au/b)}. \end{aligned} \quad (\text{B.9})$$

Substituting Eq. (B.9) into (B.6), and writing $\Gamma(u) = (u-1)!$, we obtain the thesis (B.7).

It remains to analyze the special case $u \in \{1, 2, 3, \dots\}$, $au/b = m \in \{1, 2, \dots, m-1\}$. Elementary considerations of continuity show that Eq. (B.7) holds again if one defines $\frac{\pi(1-m)_{u-1}}{\sin(\pi m)}$ as the $\varepsilon \rightarrow 0$ limit appearing in Eq. (B.8); on the other hand,

$$\begin{aligned} \frac{\pi(1-m-\varepsilon)_{u-1}}{\sin(\pi(m+\varepsilon))} &= \frac{\pi \prod_{i=0}^{u-2} (1-m-\varepsilon+i)}{\sin(\pi(m+\varepsilon))} \\ &= -\frac{\pi\varepsilon}{\sin(\pi(m+\varepsilon))} \prod_{i=0, i \neq m-1}^{u-2} (1-m-\varepsilon+i) \xrightarrow{\varepsilon \rightarrow 0} (-1)^{m+1} \prod_{i=0, i \neq m-1}^{u-2} (1-m+i), \end{aligned} \quad (\text{B.10})$$

which justifies the second equality in (B.8). \square

General results on radial, inverse Fourier transforms. Let us summarize some standard facts.

B.2 Proposition. *Let g be as in (B.1) (B.2), hence in \mathcal{L}^1 , and $f := \mathcal{F}^{-1}g \in \mathcal{L}^\infty$. f has the radial structure*

$$f(x) = F(|x|) \text{ for } x \in \mathbb{R}^d, \quad (\text{B.11})$$

where $F \in C([0, +\infty), \mathbb{C})$ is given by

$$F(\rho) := \int_0^{+\infty} d\xi \xi^{d-1} \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} G(\xi) \quad (\text{B.12})$$

$$\left(\text{intending } \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} \Big|_{\rho=0} := \lim_{s \rightarrow 0^+} \frac{J_{d/2-1}(s)}{s^{d/2-1}} = \frac{1}{2^{d/2-1}\Gamma(d/2)} \right).$$

Moreover, if $G(\xi) \geq 0$ for all $\xi \in (0, +\infty)$, then

$$\|f\|_{\mathcal{L}^\infty} = f(0) = F(0). \quad (\text{B.13})$$

Proof. The representation (B.11)(B.12) for f is well known, see, e.g., [5]. Of course, we have

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk e^{ik \bullet x} G(|k|) \quad (\text{B.14})$$

for each $x \in \mathbb{R}^d$ and, in particular,

$$f(0) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk G(|k|) ; \quad (\text{B.15})$$

if $G(\xi) \geq 0$ for all ξ , Eqs. (B.14) (B.15) imply

$$|f(x)| \leq \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dk G(|k|) = f(0) \text{ for all } x \in \mathbb{R}^d, \quad (\text{B.16})$$

yielding Eq. (B.13). \square

The Fox and Meijer functions H, G . Before computing some inverse Fourier transforms, it is necessary to say a few words on the above cited functions.

The Fox H -function [23] [36] depends on one complex variable z and on a set of parameters a_ℓ, A_ℓ ($\ell = 1, \dots, n$), a_ℓ^*, A_ℓ^* ($\ell = 1, \dots, n^*$), b_ℓ, B_ℓ ($\ell = 1, \dots, m$), b_ℓ^*, B_ℓ^* ($\ell = 1, \dots, m^*$) where n, n^*, m, m^* are nonnegative integers, $a_\ell, a_\ell^*, b_\ell, b_\ell^*$ are complex numbers and $A_\ell, A_\ell^*, B_\ell, B_\ell^*$ are real and positive. The definition reads

$$\begin{aligned} & H \left((a_1, A_1), \dots, (a_n, A_n); (a_1^*, A_1^*), \dots, (a_{n^*}^*, A_{n^*}^*) \right. \\ & \quad \left. (b_1, B_1), \dots, (b_m, B_m); (b_1^*, B_1^*), \dots, (b_{m^*}^*, B_{m^*}^*) \mid z \right) \\ & := \frac{1}{2\pi i} \int_{\mathfrak{P}} ds z^s \frac{\prod_{\ell=1}^n \Gamma(1 - a_\ell + A_\ell s) \prod_{\ell=1}^m \Gamma(b_\ell - B_\ell s)}{\prod_{\ell=1}^{n^*} \Gamma(a_\ell^* - A_\ell^* s) \prod_{\ell=1}^{m^*} \Gamma(1 - b_\ell^* + B_\ell^* s)}, \end{aligned} \quad (\text{B.17})$$

where \mathfrak{P} is an oriented path in \mathbb{C} such that the “left” and the “right” poles of the integrand in (B.17) are on the left and on the right of \mathfrak{P} , respectively. By definition, the sets of the left and right poles are

$$\mathcal{L} \equiv \mathcal{L}((a_1, A_1), \dots, (a_n, A_n)) := \quad (\text{B.18})$$

$$\{\sigma \in \mathbb{C} \mid \sigma \text{ is a pole of the function } s \mapsto \Gamma(1 - a_\ell + A_\ell s) \text{ for some } \ell \in \{1, \dots, n\}\}$$

$$= \left\{ \frac{-1 + a_\ell - k}{A_\ell} \mid \ell = 1, \dots, n; k = 0, 1, 2, \dots \right\} ;$$

$$\mathcal{R} \equiv \mathcal{R}((b_1, B_1), \dots, (b_m, B_m)) := \quad (\text{B.19})$$

$$\{\sigma \in \mathbb{C} \mid \sigma \text{ is a pole of the function } s \mapsto \Gamma(b_\ell - B_\ell s) \text{ for some } \ell \in \{1, \dots, m\}\}$$

$$= \left\{ \frac{b_\ell + k}{B_\ell} \mid \ell = 1, \dots, m; k = 0, 1, 2, \dots \right\} ;$$

the parameters $a_\ell, A_\ell, b_\ell, B_\ell$, must be such that \mathcal{L} and \mathcal{R} do not intersect.

The path \mathfrak{P} must be conveniently specified, and suitable conditions must be put on the parameters, on z and on $\arg z$ to ensure a nonambiguous definition of z^s and the convergence of the integral in Eq. (B.17). For the sake of the present paper, it suffices to consider the following choices and conditions ⁽²³⁾:

$$\mathfrak{P} := \text{any path from } c - i\infty \text{ to } c + i\infty \text{ (} c \in \mathbb{R} \text{)} \quad (\text{B.20})$$

with \mathcal{L} on its left, \mathcal{R} on its right ;

$$z \in \mathbb{C} \setminus (-\infty, 0] , \quad |\arg z| < \pi ; \quad (\text{B.21})$$

$$\Upsilon > 0 \text{ and } |\arg z| < \frac{\Upsilon\pi}{2} \text{ if } \Upsilon < 2 , \quad (\text{B.22})$$

$$\Upsilon := \sum_{\ell=1}^n A_\ell - \sum_{\ell=1}^{n^*} A_\ell^* + \sum_{\ell=1}^m B_\ell - \sum_{\ell=1}^{m^*} B_\ell^* . \quad (\text{B.23})$$

The second condition in Eq. (B.21) is understood as the definition of $\arg z$ to be used in Eq. (B.17), where one intends $z^s := e^{s(\log|z| + i\arg z)}$. Due to condition (B.22), the integrand of Eq. (B.17) decays exponentially ⁽²⁴⁾.

One could implement the definition (B.17) of H with other choices of the path \mathfrak{P} and other conditions on the parameters and on z [23], but these alternatives are not considered in this paper.

The Meijer G-function [10] [22] [28] is an H -function with $A_\ell, B_\ell, A_\ell^*, B_\ell^* = 1$ for all ℓ ; its definition reads

$$\begin{aligned} & G \left(\begin{matrix} a_1, \dots, a_n; a_1^*, \dots, a_{n^*}^* \\ b_1, \dots, b_m; b_1^*, \dots, b_{m^*}^* \end{matrix} \middle| z \right) \\ &:= \frac{1}{2\pi i} \int_{\mathfrak{P}} ds z^s \frac{\prod_{\ell=1}^n \Gamma(1 - a_\ell + s) \prod_{\ell=1}^m \Gamma(b_\ell - s)}{\prod_{\ell=1}^{n^*} \Gamma(a_\ell^* - s) \prod_{\ell=1}^{m^*} \Gamma(1 - b_\ell^* + s)} . \end{aligned} \quad (\text{B.24})$$

The path \mathfrak{P} in the above integral is chosen as in Eq. (B.20), and conditions (B.21) (B.22) are prescribed. Of course, in the present case the left and right poles \mathcal{L}, \mathcal{R}

²³Here and in the sequel the expression “ \mathfrak{P} is a path from $c - i\infty$ to $c + i\infty$ ”, with $c \in \mathbb{R}$, must be understood as follows: \mathfrak{P} has a parametrization $s = \mathbf{p}(t)$, where $\mathbf{p} \in C^1(\mathbb{R}, \mathbb{C})$ is such that $\mathbf{p}(t) = c + it + O(1/t)$ and $\mathbf{p}'(t) = i + O(1/t)$ for $t \rightarrow \pm\infty$. If this happens and ϕ, ψ are two complex functions defined on the support of \mathfrak{P} , we write “ $\phi(s) \sim \psi(s)$ for $s \rightarrow c \pm i\infty$ along \mathfrak{P} ” to mean the following: for any parametrization \mathbf{p} as before, one has $\phi(\mathbf{p}(t))/\psi(\mathbf{p}(t)) \rightarrow 1$ for $t \rightarrow \pm\infty$.

²⁴In fact, let $\phi(s)$ denote the function under the sign of integral in Eq. (B.17), also depending on z and on the parameters a_ℓ, \dots, B_ℓ^* . Using the Stirling’s formula for the Gamma function, one readily checks that, for $s \rightarrow c \pm i\infty$ along \mathfrak{P} , $|\phi(s)| \sim \Phi_\pm |\text{Im}s|^{-\omega + c\Omega} e^{-(\pi\Upsilon/2 \pm \arg z)|\text{Im}s|}$ where Υ is defined by (B.23), $\omega := \text{Re}(\sum_{\ell=1}^n a_\ell + \sum_{\ell=1}^{n^*} a_\ell^* - \sum_{\ell=1}^m b_\ell - \sum_{\ell=1}^{m^*} b_\ell^*)$, $\Omega := \sum_{\ell=1}^n A_\ell + \sum_{\ell=1}^{n^*} A_\ell^* - \sum_{\ell=1}^m B_\ell - \sum_{\ell=1}^{m^*} B_\ell^*$ and Φ_\pm are constants that could be written as well in terms of $z, a_\ell, \dots, B_\ell^*$. The coefficients $\pi\Upsilon/2 \pm \arg z$ in the exponential are both positive if and only if Υ and $\arg z$ fulfill the inequalities (B.22).

and the parameter Υ are described by Eqs. (B.18) (B.19) (B.23) with $A_\ell, \dots, B_\ell^* = 1$; in particular, the last of these equations becomes

$$\Upsilon := n - n^* + m - m^* . \quad (\text{B.25})$$

In the previous considerations it has been assumed $z \neq 0$; the H - or G - functions at $z = 0$ are defined taking the $z \rightarrow 0$ limit, if this exists.

The H - and G -functions have certain symmetry properties with respect to their parameters, which appear from the definitions (B.17) (B.24). For example, from (B.24) it is evident that the G -function is invariant under a permutation of anyone of the four sequences (a_1, \dots, a_n) , (a_1^*, \dots, a_n^*) , (b_1, \dots, b_m) , (b_1^*, \dots, b_m^*) . Moreover, a G -function does not change if we remove from the list of its parameters any pair $(a_{\ell_1}, b_{\ell_2}^*)$ with $a_{\ell_1} = b_{\ell_2}^*$, or any pair $(a_{\ell_1}^*, b_{\ell_2})$ with $a_{\ell_1}^* = b_{\ell_2}$; in fact, for any such pair there is a mutual cancellation of the corresponding Gamma functions in (B.24).

Let us repeat a comment appearing also in the main text: in comparison with the H -function, the G -function is more frequently implemented in standard packages for symbolic or numerical computations with special functions. We also mention that G can be expressed as a linear combination of generalized hypergeometric functions [22] [28]. As a final comment, the notation employed here for G is suggested by the `Mathematica` command for this function, and the notation for H arises from a natural generalization of the style used for G ⁽²⁵⁾. In the sequel we are interested in the H - or G - function in some special cases, where all the parameters and the variable are real.

Some inverse Fourier transforms.

B.3 Lemma. *Let*

$$g(k) := \frac{|k|^{2a-d}}{1 + |k|^{2b}} \quad \text{for } k \in \mathbb{R}^d \quad (b > a > 0), \quad f := \mathcal{F}^{-1}g ; \quad (\text{B.26})$$

then the following holds.

²⁵In the literature, the function H of Eq. (B.17) is more frequently written as

$$H_{pq}^{mn} \left(z \left| \begin{array}{c} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{array} \right. \right)$$

where $p := n + n^*$, $((a_1, A_1), \dots, (a_p, A_p)) := ((a_1, A_1), \dots, (a_n, A_n), (a_1^*, A_1^*), \dots, (a_n^*, A_n^*))$, $q := m + m^*$ and $((b_1, B_1), \dots, (b_q, B_q)) := ((b_1, B_1), \dots, (b_m, B_m), (b_1^*, B_1^*), \dots, (b_m^*, B_m^*))$; see, e.g., [36].

The function G of Eq. (B.24) is more frequently written as

$$G_{pq}^{mn} \left(z \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right)$$

where $p := n + n^*$, $(a_1, \dots, a_p) := (a_1, \dots, a_n, a_1^*, \dots, a_n^*)$, $q := m + m^*$, $(b_1, \dots, b_q) := (b_1, \dots, b_m, b_1^*, \dots, b_m^*)$.

(i) $g \in \mathcal{L}^1$, so that $f \in \mathcal{L}^\infty$. f has the radial structure (B.11) $f(x) = F(|x|)$ where

$$F(\rho) := \int_0^{+\infty} d\xi \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} \frac{\xi^{2a-1}}{1+\xi^{2b}} \quad \text{for } \rho \in (0, +\infty), \quad (\text{B.27})$$

$$F(0) := \frac{\pi}{2^{d/2}\Gamma(d/2)b \sin(\pi a/b)} . \quad (\text{B.28})$$

Moreover

$$\|f\|_{\mathcal{L}^\infty} = f(0) = F(0) . \quad (\text{B.29})$$

(ii) For all $\rho \in [0, +\infty)$,

$$F(\rho) = \frac{1}{2^{d/2}b} H \left(\begin{matrix} (1 - \frac{a}{b}, \frac{1}{b}); \\ (0, 1), (1 - \frac{a}{b}, \frac{1}{b}); (1 - \frac{d}{2}, 1) \end{matrix} \middle| \left(\frac{\rho}{2} \right)^2 \right) \quad (\text{B.30})$$

(note that the right hand side contains an H -function as in (B.17) with $n = 1$, $n^* = 0$, $m = 2$, $m^* = 1$; in this case Eq. (B.23) gives $\Upsilon = 2/b > 0$).

(iii) In the rational case

$$b = \frac{N}{M} , \quad N, M \in \{1, 2, 3, \dots\} \quad (\text{B.31})$$

F can be expressed as follows:

$$F(\rho) = \frac{M}{2^{d/2+M-1}\pi^{M-1}N^{d/2}} G \left(\begin{matrix} a_1, \dots, a_N; \\ b_1, \dots, b_{N+M}; b_1^*, \dots, b_N^* \end{matrix} \middle| \left(\frac{\rho}{2N} \right)^{2N} \right), \quad (\text{B.32})$$

where:

$$\begin{aligned} a_\ell &:= 1 - \frac{a}{N} - \frac{\ell-1}{M} \quad \text{for } \ell = 1, \dots, M ; \\ b_h &:= \frac{h-1}{N} \quad \text{for } h = 1, \dots, N , \quad b_{N+h} := -\frac{a}{N} + \frac{h}{M} \quad \text{for } h = 1, \dots, M ; \\ b_\ell^* &:= 1 - \frac{d}{2N} - \frac{\ell-1}{N} \quad \text{for } \ell = 1, \dots, N \end{aligned} \quad (\text{B.33})$$

(here we are considering a Meijer G -function as in Eq. (B.24) with $n = N$, $n^* = 0$, $m = N + M$, $m^* = N$; Eq. (B.25) gives $\Upsilon = 2N > 0$).

Proof. (i) All statements in this item are readily proved using Proposition B.2 with $G(\xi) = \frac{\xi^{2a-d}}{1+\xi^{2b}}$; in particular, this proposition gives

$$F(0) = \frac{1}{2^{d/2-1}\Gamma(d/2)} \int_0^{+\infty} d\xi \frac{\xi^{2a-1}}{1+\xi^{2b}} = \frac{\pi}{2^{d/2}\Gamma(d/2)b \sin(\pi a/b)} ,$$

thus justifying (B.28) (as for the integral in the above line, use Lemma B.1 with $u = 1$).

(ii) Here (and in the subsequent proof of (iii)) we assume $\rho > 0$; once Eq. (B.30) is proved for $\rho > 0$, its extension to $\rho = 0$ follows from the continuity of F at zero (and the same can be said about Eq. (B.32) of item (iii)).

Let us start from the Mellin-Barnes representation of the Bessel functions, that can be written as follows:

$$J_\nu(x) = \frac{1}{2\pi i} \int_{\mathfrak{P}} ds \left(\frac{x}{2}\right)^{\nu+2s} \frac{\Gamma(-s)}{\Gamma(1+\nu+s)} \quad (\text{B.34})$$

under the following conditions:

$$\mathfrak{P} \text{ a path from } c - i\infty \text{ to } c + i\infty \text{ (} c \in \mathbb{R} \text{)} \quad (\text{B.35})$$

such that the poles $s = 0, 1, 2, \dots$ of $\Gamma(-s)$ are on the right of \mathfrak{P} ;

$$x \in (0, +\infty), \nu \in \mathbb{C}, \operatorname{Re} \nu + 2c > 0 \quad (\text{B.36})$$

(see [29], page 115, Eq. (3.4.21) with the related comments, and [35], page 192, Eq. (7)) ⁽²⁶⁾. In the sequel we use the representation (B.34) with

$$\nu = \frac{d}{2} - 1 \geq -\frac{1}{2}, \quad c > \frac{1}{4} \quad (\text{B.37})$$

(so that $\nu + 2c > 0$); expressing in this way the term $J_{d/2-1}(\rho\xi)$ in Eq. (B.27), and reversing therein the order of the integrations in s and ξ we conclude

$$F(\rho) = \frac{1}{2\pi i} \frac{1}{2^{d/2-1}} \int_{\mathfrak{P}} ds \left(\frac{\rho}{2}\right)^{2s} \frac{\Gamma(-s)}{\Gamma(d/2+s)} \int_0^{+\infty} d\xi \frac{\xi^{2a+2s-1}}{1+\xi^{2b}}. \quad (\text{B.38})$$

On the other hand, Eq. (B.6) with $u = 1$ and a replaced by $a + s$ gives

$$\int_0^{+\infty} d\xi \frac{\xi^{2a+2s-1}}{1+\xi^{2b}} = \frac{\Gamma(a/b + s/b) \Gamma(1 - a/b - s/b)}{2b}, \quad (\text{B.39})$$

so from (B.38) we get

$$F(\rho) = \frac{1}{2\pi i} \frac{1}{2^{d/2}b} \int_{\mathfrak{P}} ds \left(\frac{\rho}{2}\right)^{2s} \frac{\Gamma(a/b + s/b) \Gamma(-s) \Gamma(1 - a/b - s/b)}{\Gamma(d/2 + s)}. \quad (\text{B.40})$$

²⁶The following remark is a variation of some considerations from page 115 of [29]. For $x \in (0, +\infty)$, $\nu \in \mathbb{C}$, $c \in \mathbb{R}$, one has $\left| \left(\frac{x}{2}\right)^{\nu+2s} \Gamma(-s) / \Gamma(1+\nu+s) \right| \sim H_{\pm} / |\operatorname{Im} s|^{\operatorname{Re} \nu + 2c + 1}$ for $s \rightarrow c \pm i\infty$ along \mathfrak{P} , for suitable constants H_{\pm} depending on x, ν, c ; therefore the integral in (B.34) is absolutely convergent under the condition $\operatorname{Re} \nu + 2c > 0$, that appears for this reason in Eq. (B.36).

Comparing this result with the general definition (B.17) of the Fox H -function we obtain the thesis (B.30), provided that the path of integration separates as required the left and right poles of the integrand. Indeed, in the case of (B.40) the sets of left and right poles are

$$\begin{aligned}\mathcal{L} &= \{\sigma \in \mathbb{C} \mid \sigma \text{ is a pole of the function } s \mapsto \Gamma(\frac{a}{b} + \frac{s}{b})\} \\ &= \{-a - bk \mid k = 0, 1, 2, \dots\};\end{aligned}\quad (\text{B.41})$$

$$\begin{aligned}\mathcal{R} &= \{\sigma \in \mathbb{C} \mid \sigma \text{ is a pole of the function } s \mapsto \Gamma(-s) \text{ or} \\ &\text{of the function } s \mapsto \Gamma(1 - \frac{a}{b} - \frac{s}{b})\} = \{0, 1, 2, \dots\} \cup \{b - a + bk \mid k = 0, 1, 2, \dots\}.\end{aligned}\quad (\text{B.42})$$

The requirement that the sets \mathcal{L}, \mathcal{R} are on the left and on the right of \mathfrak{P} is a strengthening of the conditions (B.35) initially given on this path; we can in any case choose \mathfrak{P} so as to fulfill these stronger requirements, thus getting the thesis (B.30).

(iii) We maintain for \mathfrak{P} the choice employed in the proof of (ii), and use Eq. (B.40) with $b = N/M$. After a change of variable $s \rightarrow Ns$ in the integral therein, we get

$$F(\rho) = \frac{1}{2\pi i} \frac{M}{2^{d/2}} \int_{\mathfrak{P}/N} ds \left(\frac{\rho}{2}\right)^{2Ns} \frac{\Gamma(M\frac{a}{N} + Ms) \Gamma(-Ns) \Gamma(1 - M\frac{a}{N} - Ms)}{\Gamma(\frac{d}{2} + Ns)}. \quad (\text{B.43})$$

In the above \mathfrak{P}/N indicates the path defined as follows: if $s = \mathbf{p}(t)$ is a parametrization of \mathfrak{P} , then by definition $s = \mathbf{p}(t)/N$ is a parametrization of \mathfrak{P}/N . Of course, \mathfrak{P}/N goes from $c/N - i\infty$ to $c/N + i\infty$.

Now, in the integral representation of $F(\rho)$ each Gamma function contains the term s or $-s$ multiplied by a positive integer. On the other hand, the known Gauss multiplication rule for the Gamma function [28] states that, for $n \in \{1, 2, 3, \dots\}$,

$$\Gamma(nz) = \frac{n^{nz-1/2}}{(2\pi)^{n/2-1/2}} \prod_{h=1}^n \Gamma(z + \frac{h-1}{n}); \quad (\text{B.44})$$

we use this rule with $n = M$ and $z = a/N + s$, or $n = N$ and $z = -s$, or $n = M$ and $z = 1/M - a/N - s$, or $n = N$ and $z = d/(2N) + s$, which allows to reformulate (B.43) as

$$\begin{aligned}F(\rho) &= \frac{1}{2\pi i} \frac{M}{2^{d/2+M-1}\pi^{M-1}N^{d/2}} \\ &\times \int_{\mathfrak{P}/N} ds \left(\frac{\rho}{2N}\right)^{2Ns} \frac{\prod_{h=1}^M \Gamma(\frac{a}{N} + \frac{h-1}{M} + s) \prod_{h=1}^N \Gamma(\frac{h-1}{N} - s) \prod_{h=1}^M \Gamma(-\frac{a}{N} + \frac{h}{M} - s)}{\prod_{h=1}^N \Gamma(\frac{d}{2N} + \frac{h-1}{N} + s)}.\end{aligned}\quad (\text{B.45})$$

Now, comparing with the general definition (B.24) of the G -function we obtain the thesis (B.32) (again, the path separates correctly the left and right poles of the integrand). \square

More on inverse Fourier transforms. The forthcoming statement relies on the modified Bessel function of the second kind (Macdonald function) K_μ and on the hypergeometric function ${}_2F_1$ [28] [35].

B.4 Lemma. *Let $\mu, \sigma \in \mathbb{R}$, and define*

$$g(k) := \frac{K_\mu(|k|)}{|k|^\sigma} \quad (k \in \mathbb{R}^d \setminus \{0\}) . \quad (\text{B.46})$$

Then $g \in \mathcal{L}^1$ if and only if

$$|\mu| + \sigma < d . \quad (\text{B.47})$$

Assuming this,

$$f := \mathcal{F}^{-1}g \quad (\text{B.48})$$

is in \mathcal{L}^∞ and possesses the radial structure (B.11) $f(x) = F(|x|)$, where

$$F(\rho) := \frac{\Gamma(d/2 + \mu/2 - \sigma/2) \Gamma(d/2 - \mu/2 - \sigma/2)}{2^{\sigma+1-d/2} \Gamma(d/2)} \quad (\text{B.49})$$

$$\times {}_2F_1(d/2 + \mu/2 - \sigma/2, d/2 - \mu/2 - \sigma/2; d/2; -\rho^2) \quad \text{for } \rho \in [0, +\infty) .$$

In particular, if $\sigma = \mu$,

$$F(\rho) = \frac{2^{d/2-\mu-1} \Gamma(d/2 - \mu)}{(1 + \rho^2)^{d/2-\mu}} \quad \text{for } \rho \in [0, +\infty) . \quad (\text{B.50})$$

Moreover

$$f \in \mathcal{L}^2 \Leftrightarrow g \in \mathcal{L}^2 \Leftrightarrow 2(|\mu| + \sigma) < d . \quad (\text{B.51})$$

Proof. We apply Proposition B.2 with $G(\xi) := K_\mu(\xi)/\xi^\sigma$. Recalling that $K_\mu(\xi) = O(e^{-\xi}/\sqrt{\xi})$ for $\xi \rightarrow +\infty$ and that $K_\mu(\xi) \sim \text{const.}\xi^{-|\mu|}$ for $\mu \neq 0$ and $\xi \rightarrow 0^+$, $K_0(\xi) \sim -\ln \xi$ for $\xi \rightarrow 0^+$, we see that $\int_0^{+\infty} d\xi \xi^{d-1} |G(\xi)| < +\infty$, i.e., $g \in \mathcal{L}^1$, if and only if (B.47) holds. Assuming this the general rule (B.11) for radial, inverse Fourier transforms gives in the present case $\mathcal{F}^{-1}g(x) = F(|x|)$ with

$$F(\rho) := \int_0^{+\infty} d\xi \xi^{d-\sigma-1} \frac{J_{d/2-1}(\rho\xi)}{(\rho\xi)^{d/2-1}} K_\mu(\xi) \quad \text{for } \rho > 0, \quad F(0) := \lim_{\rho \rightarrow 0^+} F(\rho) . \quad (\text{B.52})$$

The integral in (B.52) can be computed using results from [35], §13.45, page 410; Eq.(1) therein yields our Eq. (B.49) and Eq.(2) therein (in a variant taking into account the identity $K_{-\mu} = K_\mu$) yields our Eq. (B.50).

To conclude, let us justify the statements in (B.51). Since $f = \mathcal{F}^{-1}g$, it is obvious that $f \in \mathcal{L}^2 \Leftrightarrow g \in \mathcal{L}^2$. On the other hand,

$$g \in \mathcal{L}^2 \Leftrightarrow \int_0^{+\infty} d\xi \xi^{d-1} \left(\frac{K_\mu(\xi)}{\xi^\sigma} \right)^2 < +\infty \Leftrightarrow 2(|\mu| + \sigma) < d . \quad (\text{B.53})$$

The first equivalence is obvious; to obtain the second equivalence note that, by the previously mentioned asymptotic behavior of K_μ , the function of ξ in the above integral decays exponentially for $\xi \rightarrow +\infty$ and behaves like $1/\xi^{2(|\mu|+\sigma)-d+1}$ for $\mu \neq 0$ and $\xi \rightarrow 0^+$, or like $(\ln \xi)^2/\xi^{2\sigma-d+1}$ for $\mu = 0$ and $\xi \rightarrow 0^+$. \square

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